STAT 134 - Instructor: Adam Lucas

Practice Midterm 2 problems

SOLUTIONS

Monday, November 16, 2020

Print your name: ______________________________

SID Number: ______________________________

Exam Information and Instructions:

- You will have 52 minutes to take this exam. Closed book/notes/etc. No calculator or computer.
- We will be using Gradescope to grade this exam. Write any work you want graded on the front of each page, in the space below each question. Additionally, write your SID number in the top right corner on every page.
- Provide calculations or brief reasoning in every answer.
- Unless stated otherwise, you may leave answers as unsimplified numerical and algebraic expressions, and in terms of the Normal c.d.f. Φ. Finite sums are fine, but simplify any infinite sums.

GOOD LUCK!

SID: ________________________

1. Beloved Stat 134 students Jonny and Ethan have very different study habits for the final. Jonny quits studying at a constant hazard rate $\lambda = \frac{1}{2}$ (i.e. average 2 hours). Ethan randomly quits anytime between 0 and 4 hours (also average 2 hours). You may assume Jonny and Ethan study independently. What is the chance the first quitter stops studying between 2 and 3 hours.

   Solution:

   Let $T_J \sim exp(\frac{1}{2})$ equal the time until Jonny quits and $T_E \sim U(0,4)$ equal the time until Ethan quits. Let $T = min(T_J, T_E)$. The cdf of T is

   $$F(t) = 1 - P(T_E > t)P(T_J > t) = 1 - \left(\frac{4-t}{4}\right)(e^{-t/2}) = 1 - e^{-t/2} + \frac{te^{-t/2}}{4}$$

   for $0 < t < 4$. The answer is $F(3) - F(2) = e^{-1}/2 - e^{-3/2}/4$.

2. For each distribution of X indicated in the parts below, evaluate $Var(X)$ as a simple fraction,without involving the constants c_1, c_2, c_3.

   (a) X with values in $[0,1]$ and density $f_X(x) = c_1 x^2(1-x)^3$ for $0 < x < 1$.

   (b) X with values in $[0,\infty)$ and density $c_2 x^4 e^{-3x}$ for $0 \le x < \infty$

   Solutions:

   (a) $X \sim Beta(r = 3, s = 4)$ so $Var(X) = \frac{rs}{(r+s)^2(r+s+1)} = 3/98$

   (b) $X \sim Gamma(r = 5, \lambda = 3$ so $Var(X) = \frac{r}{\lambda^2} = 5/9$

3. The joint density of X and Y is

   $$f(x,y) = \begin{cases} \frac{4y}{x} & \text{for } 0 < x < 1 \text{ and } 0 < y < x \\ 0 & \text{otherwise} \end{cases}$$

   Find the following:

   (a) $E(XY)$

   (b) The marginal density of X

   Solutions:

   (a) $E(XY) = 1/3$

   (b) $f_X(x) = 2x$

4. Let U follow a Uniform $(0,1)$ distribution. Find the distribution of $-log(U)$.

   Solution:

   Let $S = -log(U) \Rightarrow U = e^{-S} \Rightarrow \frac{ds}{du} = -\frac{1}{U} \Rightarrow |\frac{ds}{du}| = \frac{1}{u}$

$$\begin{aligned} f_S(s) &= \frac{f_U(u)}{|\frac{ds}{du}|} \\ &= \frac{1}{\frac{1}{u}} = \frac{1}{\frac{1}{e^{-s}}} \\ &= e^{-s} \Rightarrow Exp(1) \end{aligned}$$

5. 10 friends plan to arrive to lunch at noon. The arrival time of each friend is uniform from 11:45 am to 12:15 pm and independent of everyone else.

   (a) How many minutes after 11:45 is the 3rd earliest person expected to arrive?

   (b) What is the variance of the arrival time of the 3rd earliest person?

   (c) Find chance that 3 friends arrive before noon, 5 friends arrive between noon and 12:10, and the last 2 friends arrive between 12:10 and 12:15.

   (d) Find the chance that all 10 friends arrive within 10 minutes of one another.

   (a) The number of minutes that each arrival is after 11:45, X_i, can be expressed as $30U_i$, where $U_i \sim Unif(0,1)$.
   $U_{(3)} \sim Beta(3,8)$, so, $E(U_{(3)}) = 3/11$, and $E(X_{(3)}) = 30 \times 3/11 = 90/11$. The 3rd earliest person is expected to arrive 90/11 minutes after 11:45.

   (b)
   $$Var(X_{(3)}) = 30^2 Var(U_{(3)}) = 30^2 \frac{3 \cdot 8}{(3+8)^2(3+8+1)} = 30^2 \frac{2}{121}$$
   .

   (c) Using the multinomial distriution this chance is:
   $$\binom{10}{3,5,2} \cdot \left(\frac{15}{30}\right)^3 \cdot \left(\frac{10}{30}\right)^5 \cdot \left(\frac{5}{30}\right)^2$$

   (d) This chance can be formulated as follows:
   $$P(X_{(10)} - X_{(1)} < 10) = P(30U_{(10)} - 30U_{(1)} < 10) = P(U_{(10)} - U_{(1)} < 1/3)$$
   . Let $X = U_{(10)} - U_{(1)}$. Recall that
   $$U_{(10)} - U_{(1)} \sim Beta(1, 11)$$
   .
   So $P(X < 1/3) = 1 - P(X > 1/3) = 1 - (2/3)^{10}$, and this is our answer.

6. $X \sim \mathcal{N}(0,1)$ the standard normal distribution.

   (a) What is $\mathbb{E}X^3$?

(b) What is $\mathbb{E}X^4$?

(a) By symmetry, 0.

(b) We simply need to calculate the MGF of a standard normal distribution.

$$\begin{aligned}\mathbb{E}e^{tX} &= \int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-(x-t)^2/2+t^2/2}dx \\ &= e^{t^2/2}\end{aligned}$$

The 4-th derivative at 0 of the MGF gives $\mathbb{E}X^4 = 3$.