Statistics 134 - Instructor: Adam Lucas

Final Exam Solutions

Tuesday, May 14, 2019

Print your name: ________________________________

SID Number: ________________________________

Exam Information and Instructions:

- You will have 170 minutes to take this exam. Closed book/notes/etc. No calculator or computer.
- We will be using Gradescope to grade this exam. Write any work you want graded on the front of each page, in the space below each question. Additionally, write your SID number in the top right corner on every page.
- Please use a dark pencil (mechanical or #2), and bring an eraser. *If you use a pen and make mistakes, you might run out of space to write in your answer.*
- Provide calculations or brief reasoning in every answer.
- Unless stated otherwise, you may leave answers as unsimplified numerical and algebraic expressions, and in terms of the Normal c.d.f. Φ. Finite sums are fine, but simplify any infinite sums.
- Do your own unaided work. Answer the questions on your own. The students around you have different exams.

I certify that all materials in the enclosed exam are my own original work.

Sign your name: ________________________________

GOOD LUCK!

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1. (15 pts) Let X, Y have joint density given by

$$f_{X,Y}(x,y) = \frac{\lambda}{y}e^{-\lambda y}, \quad 0 < x < y.$$

(a) (5 pts) Find the marginal distribution of Y.

(b) (5 pts) Find the conditional distribution of X, given $Y = y$. You should recognize this as one of our named distributions.

(c) (5 pts) Use (a) and (b) to find $\mathbb{E}(X)$.

2. (15 pts) Brian and Yiming play a game, where Brian and Yiming each draws five numbers without replacement from separate sets $\{1, 2, \ldots, 100\}$.

   (a) (5 pts) What is the expected number of common numbers in their choices?

(b) (5 pts) What is the variance of the number of common numbers in their choices?

(c) (5 pts) What is the probability that Brian and Yiming select at least one number in common?

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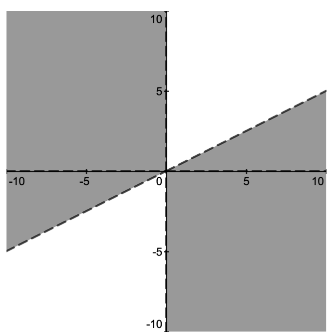
3. (15 pts) Let X, Y be independent standard normal.

   (a) (5 pts) Find $P(X < kY)$.

   (b) (10 pts) It can be shown that $Z = Y/X$ follows the Cauchy distribution. Use rotational symmetry of X, Y to find $P(Z < k)$, for $k > 0$, *without* directly using the CDF of the Cauchy distribution given on your reference sheet. Hint: the region $Y/X < k$ is shaded in the graph below, for $k = 0.5$.



4. (15 pts) Consider $X \sim \text{Beta}(r, s)$. We have not yet seen the MGF of a Beta; let us now derive the result. This requires a slightly different approach than we have seen before, as the result is not so elegant as other MGFs we have seen.

   (a) (5 pts) Calculate $E(X^k)$, where k is a positive integer. You may leave your result in terms of $\Gamma(\cdot)$.

   (b) (5 pts) Let $b_k = E(X^k)$ denote the result you have calculated above. Use the Taylor series for e^x to write $M_X(t)$ as a (infinite) summation containing these b_k terms. Do not attempt to simplify this result!

(c) (5 pts) Now we must check, is this MGF defined? Give simple lower and upper bounds for all the b_k terms. Use this to give lower and upper bounds for $M_X(t)$, for $t > 0$. (Hint: think about the possible values of X^k. Your bounds should not depend on k.)

5. (15 pts) Let $X_1, X_2, \ldots, X_n$ be independent Poisson random variables with respective parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $Y = \sum_{i=1}^{n} X_i$. Find the distribution of $X_1 \mid Y = k$ and determine its expectation.

6. (12 pts)

(a) (2 pts) State Markov's inequality.

(b) (5 pts) Let Y be a random variable with moment-generating function M_Y. Use Markov's inequality to prove that, for any $t \in \mathbb{R}$,

$$\mathbb{P}(Y \geq a) \leq e^{-t \cdot a} M_Y(t).$$

(c) (5 pts) Let X be a continuous, nonnegative random variable such that $E(X) = \mu$. Consider its median m_X; recall that m_X is the value such that $P(X \leq m_X) = P(X \geq m_X)$. In terms of μ, what is the largest possible value of m_X?

7. (12 pts)

   (a) (6 pts) A family is getting ready for their trip to Yosemite. Each person is in their room, packing their bags. For each person, the time it takes them to pack their bag is exponentially distributed and independent of the time it takes any other person. On average, it takes each parent 1 hour and each child 2 hours to get ready. In a family with 2 parents and 4 children, what is the probability that it takes the family more than 2 hours to get ready?

   (b) (6 pts) After setting up the tents the family wants to rent bicycles. The rental place does only have one bicycle when they get there. Assume that the bikes being returned follow a Poisson process with rate $\frac{1}{10}$ and that nobody else is waiting for bikes. Give **two** equivalent expressions for the probability that the family will have to wait more than 30 minutes before they can start their biking tour. (No need to simplify!)

8. (16 pts) Let X, Y be random variables such that for $x, y \in \{1, 2, 3\}$,

$$\mathbb{P}(X = x, Y = y) = \frac{x + y}{36}.$$

(a) (4 pts) Give an expression for $\mathbb{P}(X = Y)$.

(b) (4 pts) Using (a), find an expression for $\mathbb{P}(X < Y)$. (Hint: Use symmetry.)

(c) (4 pts) Are X and Y independent?

(d) (4 pts) What is the probability mass function of $Z = X + Y$?

9. (10 pts) Let $X \sim$ Uniform $(-1, 1)$ (this is a continuous uniform random variable).

   (a) (5 pts) Compute the density of $Y = e^X$.

   (b) (5 pts) Let now X_1, X_2 be i.i.d. uniform random variables, and for each $i = 1, 2$, let $Y_i = e^{X_i}$. What is the joint density of $Y_{(1)}$ and $Y_{(2)}$, the minimum and the maximum of the Y_i's?

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10. (15 pts) Let X and Y be the heights (in centimeters) of mothers and daughters in a family. We know that

$$(X, Y) \sim BN(\mu_X = 170, \mu_Y = 170, \sigma_X^2 = 5^2, \sigma_Y^2 = 5^2, \rho = .8),$$

where BN stands for Bivariate Normal.

(a) (6 pts) What is the distribution of $Y - X$? Explain fully with parameters.

(b) (3 pts) Find the probability that the daughter is more than 7 cm taller than her mother.

(c) (6 pts) Consider X_1, X_2 to be the heights of two mothers selected at random from the population (so X_1 and X_2 are independent.) Let $V = X_1 - X_2$ and $W = 3X_1 - 2X_2$. Find $Corr(V, W)$.

Scratch Work:

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