Final Review Sheet Answers

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The proofs/calculations of most exercises here are omitted. Again, refer to your notes if unsure; all of the theoretical results have been discussed in lecture notes or in the textbook.

1 Fundamental Techniques

1. *Rules of probability:*

$$\text{Conditional probability/Bayes' Rule: } P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B|A)}{P(B)}$$

$$\text{DeMorgan's Laws: } P(\{A \cup B\}^c) = P(A^c \cap B^c),$$

$$P(\{A \cap B\}^c) = P(A^c \cup B^c) = P(A^c) + P(B^c) - P(A^c B^c)$$

2. *Constant and variable parts of densities:*

   $X \sim$ Gamma $(4, 6)$; $\int_0^\infty 5x^3 e^{-6x} dx = 5(\frac{\Gamma(4)}{6^4})$

3. *Independence and dependence:*

   To check that two events A, B are independent, we might check that $P(AB) = P(A)P(B)$, or equivalently, $P(A) = P(A|B)$.

   For two variables to be independent, in the discrete case, we must show that $P(X = x, Y = y) = P(X = x)P(Y = y)$. In the continuous case, we may show that $f_X(x, y) = f_X(x)f_Y(y)$, **and** that the support of X and Y do not depend on each other (i.e., for all (x, y), we have that $f_{X,Y}(x, y) > 0$ if and only if $f_X(X) > 0$ and $f_Y(y) > 0$.)

   An easy way to show that two variables are dependent is to find x, y such that $P(Y = y) > 0$ but $P(Y = y|X = x) = 0$. The idea is to find a value of X that makes a particular value of Y impossible. (A similar result holds in the continuous case.)

4. *Working with densities:*

   (a) The CDF of X is given by $F_X(x) = \int_{-\infty}^x f_X(t)dt$. By the fundamental theorem of calculus, it follows that $f_X(x) = F'_X(x)$.

   (b) $f_X(x) = \int_{-\infty}^\infty f_{X,Y}(x, y)dy$

   (c) $f_{X|Y}(x|y) = f_{X,Y}(x, y)/f_Y(y)$. Note this does not necessarily require the marginal density of X, f_X, to calculate!

2 Relationships Between Distributions

1. cX, where $X \sim$ Exp (λ), $c > 0$:

   Exp $(\frac{\lambda}{c})$. The methods we have used include using the CDF of X, using the change of variable method, and using the MGF of X and properties of MGFs.

2. Consider $X = U_{(3)}$ and $Y = U_{(7)}$, the 3rd and 7th order statistics of 10 iid Unif $(0, 1)$ RVs. What is the distribution of X/Y?

   We proved in lecture that for this type of example (ratio of joint Uniform (0,1) order statistics, where the numerator is the smaller one), the resulting distribution is a Beta. In particular, this example follows the Beta (3,4) distribution.

3. $X^2 + Y^2$, where X, Y are independent standard Normal. What is $\sqrt{X^2 + Y^2}$?

   Exp $(\frac{1}{2})$; standard Rayleigh

4. $2X + 3Y$, where X, Y independent Normal (μ, σ^2). What if X, Y are bivariate normal with correlation $\rho = 0.6$?

   $2X + 3Y \sim \mathcal{N}(5\mu, 13\sigma^2)$ if $\rho = 0$; $2X + 3Y \sim \mathcal{N}(5\mu, (13 + 7.2)\sigma^2)$ if $\rho = 0.6$

5. Consider each of the following common discrete distributions: Poisson, Binomial, Geometric, and Hypergeometric. For which of these is the sum of two independent RVs a known distribution? Under what conditions?

   Excluding the degenerate cases (e.g., $p = 0$ or $p = 1$), this holds for the first 3 distributions. For Binomial, the p values must be the same, in which case the n's are added; for the Geometric the result is Negative Binomial provided the p values are the same.

3 Symmetry

1. Under some conditions, we can quickly recognize the expectation of a random variable X to be zero. What are they?

   The distribution/density of X must be symmetric about the origin, and $\mathbb{E}(|X|) < \infty$; i.e. the expectation must be defined.

2. Find the probability that the last ace in a standard, well-shuffled deck is at position 47 or greater.

   Using symmetry between the front and back of the deck, this answer is

$$1 - P(\text{no aces in first 6}) = 1 - \frac{\binom{4}{0}\binom{48}{6}}{\binom{52}{6}}$$

3. You and I each roll a fair n sided die. Without using a summation, find the probability your roll is strictly greater than mine.

   $P(X < Y) = P(X > Y)$, and $P(X < Y) + P(X > Y) + P(X = Y) = 1$. Therefore,

$$P(X > Y) = \frac{1 - \frac{1}{n}}{2}$$

4. Let X, Y be independent $\mathcal{N}(0, \sigma^2)$ random variables. Without using Φ, find $P(X + 2Y > 0, X > 0)$. What is the reason behind this answer?

   Using the rotational symmetry of the joint distribution of (X, Y),

$$P(X + 2Y > 0, X > 0) = \frac{\frac{\pi}{2} + \arctan(\frac{1}{2})}{2\pi}$$

5. Let $X_1, X_2, \ldots, X_n$ be identically distributed, exchangeable variables such that their sum is always a constant. Find $Corr(X_i, X_j)$ for $i \neq j$.

   $Corr(X_i, X_j) = -1/(n - 1)$.

4 To Infinity, and Beyond

1. Let $X \sim$ Geom (p) on $\{1, 2, \ldots\}$. It is not easy to directly find $\mathbb{E}(X)$ using the formula $\sum_{x=1}^{\infty} xP(X = x)$. We have shown three alternate methods for finding this expectation; what are they?

   The three methods are (i) the tail sum formula for expectations of discrete RVs, (ii) using the MGF of a Geometric, and (iii) conditioning on the result of the first toss.

2. Two players simultaneously toss coins which land heads with probabilities p_1 and p_2 respectively. They continue until exactly one player's coin lands heads; that player is the winner. Show that the probability Player 1 wins is $\frac{p_1 q_2}{p_1 q_2 + q_1 p_2}$.

   Hint: write this out as a sequence of tosses. For the first player to win on the n^{th} toss, what has to happen?

3. Find $\mathbb{E}(X(X-1))$ for $X \sim$ Pois (μ).

   Using the function rule, we observe that this value is μ^2.

4. Let Y have density $f_Y(y) = \frac{\lambda}{2} e^{-\lambda|y|}$, for $y \in \mathbb{R}$. With little computation, find $\mathbb{E}(|Y|)$ and $\mathbb{E}(Y)$.

   If we look at the graph of the density of Y, or through the change-of-variable formula, we observe that $|Y| \sim$ Exp (λ). Thus $\mathbb{E}(|Y|) = \frac{1}{\lambda}$, and $\mathbb{E}(Y) = 0$ by symmetry.

5 Could You Rephrase That?

1. Suppose there are $3n$ people in a room, divided into groups of 3. What is the chance that there is at least one group where two members share the same birthday?

$$1 - \left(\frac{364 \cdot 363}{365^2}\right)^n$$

2. Continued from (a): Approximate this chance for large n.

   We use a Poisson approximation, with $p = P(\text{shared b-day in group}) = 1 - \frac{364 \cdot 363}{365^2}$. Then this probability from (a) is approximately $1 - e^{-np}$ for large n.

3. Let $X \sim$ Gamma (r, λ), where r is an integer. Use what we know about the Poisson process to obtain the CDF of X.

$$\begin{aligned} P(X < t) &= P(N_t \geq r) \\ &= 1 - P(N_t < r) \\ &= 1 - \sum_{k=0}^{r-1} e^{\lambda t} \frac{(\lambda t)^k}{k!} \end{aligned}$$

4. Let X be an arbitrary random variable with an invertible CDF F_X. What is the random variable formed by $F_X(X)$?

   Let $Z = F_X(X)$. Then,

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P(F_X(X) \leq z) \\ &= P(F_X^{-1}(F_X(X)) \leq F_X^{-1}(z)) \\ &= P(X \leq F_X^{-1}(z)) \\ &= F_X(F_X^{-1}(z)) \\ &= z, \quad z \in [0, 1] \end{aligned}$$

   We conclude that $Z \sim$ Unif (0,1).

6 Some Useful Results

1. Let $X \sim \text{Pois}\ (\mu)$, $Y \sim \text{Pois}\ (\lambda)$, $X \perp Y$. What is the distribution of $X+Y$? (Hint: use the binomial theorem, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.) $X + Y \sim \text{Pois}\ (\mu + \lambda)$

2. Use Markov's Inequality to derive Chebyshev's Inequality. See pg. 191 – 192 in Pitman's *Probability*.

3. Let $X \sim \text{Exp}\ (\lambda_X)$, $Y \sim \text{Exp}\ (\lambda_Y)$. What is $P(X < Y)$? $\frac{\lambda_X}{\lambda_X + \lambda_Y}$

4. Find the distribution of $\min\{X_1, X_2, \ldots, X_n\}$, where the X_i's are independent $\text{Exp}\ (\lambda)$ variables.

   Let M denote the minimum. Then $M \sim \text{Exp}\ (n\lambda)$. Note this result generalizes to the case where the rates are all different; you simply add the rate parameters.

5. Suppose cars and trucks arrive at a bridge according to independent Poisson processes with rates λ_c and λ_r per minute respectively. Given that n vehicles arrive in t minutes, what is the distribution of $N_{c,t}$, the number of cars to arrive by time t?

   The easiest way to proceed here is using the conditional probability rule. We find that $N_{c,t}|N_t = n \sim \text{Binom}\ (n, \frac{\lambda_c}{\lambda_c + \lambda_r})$.