Stat 134: Conditional Probabilities, Distributions, & Expectations Review

Solutions

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Problem 1

Let $X_1 \sim \text{Geom}(p_1)$, $X_2 \sim \text{Geom}(p_2)$, $X_1 \perp X_2$, both on $\{1, 2, \ldots\}$. Find:

a. $P(X_1 \leq X_2)$;

b. $P(X_1 = x \mid X_1 \leq X_2)$. Recognize $X_1 | X_1 \leq X_2$ as a named distribution, and state the parameter(s).

a) $$\mathbb{P}(X_1 \leq X_2) = \sum_{k=1}^{\infty} \mathbb{P}(X_1 = k, X_2 \geq k)$$
$$= \sum_{k=1}^{\infty} q_1^{k-1} p_1 q_2^{k-1}$$
$$= p_1 \sum_{k=1}^{\infty} (q_1 q_2)^{k-1} = \boxed{\frac{p_1}{1 - q_1 q_2}.}$$

b) We proceed using the conditional prob. rule:

For $x \in \mathbb{N}$,
$$\mathbb{P}(X_1 = x \mid X_1 \leq X_2) = \frac{\mathbb{P}(X_1 = x, X_1 \leq X_2)}{\mathbb{P}(X_1 \leq X_2)}$$
$$= \frac{\mathbb{P}(X_1 = x)\,\mathbb{P}(X_2 \geq x)}{\mathbb{P}(X_1 \leq X_2)}$$
$$= \frac{\left(q_1^{x-1} \cancel{p_1}\right)\left(q_2^{x-1}\right)}{\left(\frac{\cancel{p_1}}{1 - q_1 q_2}\right)} = (q_1 q_2)^{x-1}(1 - q_1 q_2)$$

$\therefore$, $\boxed{X_1 | X_1 \leq X_2 \sim \text{Geom}(1 - q_1 q_2)}$. (on $\{1, 2, \ldots\}$)

Problem 2

Let $Y \sim \text{Beta}\,(r, s)$. Conditioned on $Y = y$, let $X \sim \text{Geometric}\,(y)$ on $\{0, 1, 2, \ldots\}$ For simplicity, assume $r, s > 1$.

a. What is $\mathbb{E}(X \mid Y = y)$?

b. Find $\mathbb{E}(X)$.

c. Find $P(X = x)$, for $x \in \{0, 1, 2, \ldots\}$.

a) $\mathbb{E}(X \mid Y = y) = \dfrac{1-y}{y} = \dfrac{1}{y} - 1$

b) $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X \mid Y))$ (law of iterated expectations)

$= \mathbb{E}\left(\dfrac{1}{Y} - 1\right) = \mathbb{E}\left(\dfrac{1}{Y}\right) - 1$, where

$$\mathbb{E}\left(\frac{1}{Y}\right) = \int_0^1 \frac{1}{y} \cdot \frac{1}{B(r,s)}\, y^{r-1}(1-y)^{s-1}\, dy$$

$$= \frac{1}{B(r,s)} \cdot B(r-1,s) \underbrace{\int_0^1 \frac{1}{B(r-1,s)} \cdot y^{r-2}(1-y)^{s-1}\, dy}_{=1}$$

$$= \frac{B(r-1,s)}{B(r,s)}.$$

$\therefore$, $\boxed{\mathbb{E}(X) = \dfrac{B(r-1,s)}{B(r,s)} - 1}$, where

$$B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$$

c) To find $\mathbb{P}(X=x)$, we use the law of total prob.:

$$\mathbb{P}(X=x) = \int_0^1 \mathbb{P}(Y\in dy)\,\mathbb{P}(X=x \mid Y=y)$$

$$= \int_0^1 f_Y(y)(1-y)^x (y)\,dy$$

$$= \int_0^1 \frac{1}{B(r,s)}\, y^{r-1}(1-y)^{s-1}(1-y)^x (y)\,dy$$

$$= \frac{1}{B(r,s)}\, B(r+1,s+x) \int_0^1 \frac{1}{B(r+1,s+x)}\, y^r (1-y)^{s+x-1}\,dy$$

$$= \boxed{\frac{B(r+1,s+x)}{B(r,s)}}$$, where $B(r,s)$ is defined as above.

Feel free to simplify this result if you like; for the intrepid, you may find that the result has a special form if r,s are integers. It will resemble the negative hypergeometric distribution, though the underlying reason is unclear.

Problem 3

Suppose a proportion p of a population has a gene m that makes them predisposed to migraines. Of these people, the number of migraines they experience in a year follows a Poisson process with rate μ per year, whereas the rest of the population experiences migraines according to a Poisson process with rate λ.

a. What is the probability that a randomly selected individual experiences no migraines in a given year?

b. Let N_t denote the number of migraines a randomly selected individual experiences in t years. Find $\mathbb{E}(N_t)$.

c. Find $Var(N_t)$.

Let $\mathbb{I}_m$ be the indicator that an individual has gene m. The process here is to condition on $\mathbb{I}_m$.

Observe given $\mathbb{I}_m = 1$, $N_t|_{\mathbb{I}_m=1} \sim \text{Pois}(\mu t)$,
& given $\mathbb{I}_m = 0$, $N_t|_{\mathbb{I}_m=0} \sim \text{Pois}(\lambda t)$.

a) $$\mathbb{P}(N_1=0) = \sum_{k=0}^{1} \mathbb{P}(\mathbb{I}_m=k)\mathbb{P}(N_t=0 \mid \mathbb{I}_m=k)$$
$$= p(e^{-\mu\cdot 1}) + q(e^{-\lambda\cdot 1}) = \boxed{pe^{-\mu} + qe^{-\lambda}.}$$

b) $$\mathbb{E}(N_t) = \mathbb{E}(\mathbb{E}(N_t \mid \mathbb{I}_m))$$
$$= \sum_{k=0}^{1} \mathbb{E}(N_t \mid \mathbb{I}_m=k)\mathbb{P}(\mathbb{I}_m=k)$$
$$= \boxed{p(\mu t) + q(\lambda t).}$$

c) We use our rule for variance by conditioning:

$$\mathrm{Var}(N_t) = \mathbb{E}(\mathrm{Var}(N_t \mid \mathbb{I}_m)) + \mathrm{Var}(\mathbb{E}(N_t \mid \mathbb{I}_m))$$

Examine each term individually:

- $\mathbb{E}(\mathrm{Var}(N_t \mid \mathbb{I}_m)) = \sum_{k=0}^{1} \mathrm{Var}(N_t \mid \mathbb{I}_m = k)\,\mathbb{P}(\mathbb{I}_m = k)$

$$= p((\mu t)) + q(\lambda t) \leftarrow$$

For $X \sim \mathrm{Pois}(\mu)$, $\mathrm{Var}(X) = \mu$.

- $\mathrm{Var}(\mathbb{E}(N_t \mid \mathbb{I}_m))$: $\mathbb{E}(N_t \mid \mathbb{I}_m)$ is a RV that is a function of $\mathbb{I}_m$. It has two poss. values:

$$\mathbb{E}(N_t \mid \mathbb{I}_m) = \begin{cases} \mu t & \text{with prob. } p \\ \lambda t & \text{with prob. } q \end{cases}$$

We use our definition of $\mathrm{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$:

$$\mathrm{Var}(\mathbb{E}(N_t \mid \mathbb{I}_m)) = \mathbb{E}\Big(\big(\mathbb{E}(N_t \mid \mathbb{I}_m) - \underbrace{\mathbb{E}(\mathbb{E}(N_t \mid \mathbb{I}_m))}_{= \mathbb{E}(N_t)}\big)^2\Big)$$

$$= p(\mu t - \mathbb{E}(N_t))^2 + q(\lambda t - \mathbb{E}(N_t))^2,$$

where $\mathbb{E}(N_t)$ is given in (b).

$$\therefore \boxed{\mathrm{Var}(N_t) = p(\mu t) + q(\lambda t) + p(\mu t - \mathbb{E}(N_t))^2 + q(\lambda t - \mathbb{E}(N_t))^2}$$

Problem 4

Let X, Y have joint density $f_{X,Y}(x,y) = 2\lambda^2 e^{-\lambda(x+y)}, 0 < x < y$. It can be shown that $f_X(x) = 2\lambda e^{-2\lambda x}, x > 0$. Find:

a. The conditional density of Y, given $X = x$;

b. $\mathbb{E}(Y|X = x)$.

a) $$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2\lambda^2 e^{-\lambda x} e^{-\lambda y}}{2\lambda e^{-2\lambda x}}$$

$$= \lambda \frac{e^{-\lambda y}}{e^{-\lambda x}}, \quad 0 < x < y.$$

A good check: does this cond'l. density integrate to 1?
Try for yourself.

b) $$\mathbb{E}(Y|X=x) = \int_x^\infty y f_{Y|X}(y|x)\, dy$$

$$= \int_x^\infty \frac{\lambda y e^{-\lambda y}}{e^{-\lambda x}}\, dy$$

u-sub: $u = y - x$

$$= \frac{1}{e^{-\lambda x}} \int_x^\infty \lambda y e^{-\lambda y}\, dy$$

$$= \frac{1}{e^{-\lambda x}} \int_0^\infty \lambda (u+x) e^{-\lambda(u+x)}\, du$$

$$= \frac{\cancel{e^{-\lambda x}}}{\cancel{e^{-\lambda x}}} \int_0^\infty \lambda(u+x) e^{-\lambda u}\, du$$

$= \mathbb{E}(Z + x)$ for $Z \sim \mathrm{Exp}(\lambda)$

$$= \boxed{\lambda + x.}$$