Stat 134: Joint Distributions Review - Solutions

Adam Lucas

December 5th, 2018

Conceptual Review

Suppose X, Y are random variables with joint distribution $f_{X,Y}$ over the region $\{(x,y) \in \mathbb{R}^2 : 0 < x < y\}$.

a. Are X, Y independent? No, because $\mathbb{P}(Y<1)>0$, but $\mathbb{P}(Y<1 \mid X>1)=0$.

b. Set up an integral to find each of the following:

  i. $f_X(x)$;
  ii. $F_Y(y)$;
  iii. $P(Y < X+5)$;
  iv. $\mathbb{E}(X)$;
  v. $\mathbb{E}(g(X,Y))$.

i) $f_X(x) = \int_x^\infty f_{X,Y}(x,y)\,dy$

ii) $F_Y(y) = \int_0^y\int_0^y f_{X,Y}(x,z)\,dx\,dz$ dummy variable

iii) $\int_0^\infty\int_x^{x+5} f(x,y)\,dy\,dx$

iv) $\int_0^\infty\int_x^\infty x f_{X,Y}(x,y)\,dy\,dx$ OR $\int_0^\infty x f_X(x)\,dx$

v) $\int_0^\infty\int_0^y g(x,y) f_{X,Y}(x,y)\,dx\,dy$

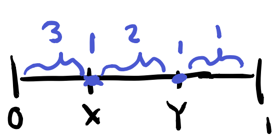
Problem 1

Based on each of the joint densities below, with minimal calculation, identify the marginal distribution of X and Y.

a. $f_{X,Y}(x,y) = 3360x^3(y-x)^2(1-y),\ \ 0 < x < y < 1$ (Bonus: how did we compute the constant of 3360?);

b. $f_{X,Y}(x,y) = \lambda^3 e^{-\lambda y}(y-x),\ \ 0 < x < y$;

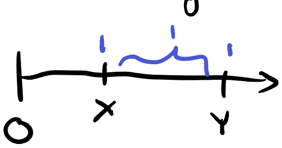
c. $f_{X,Y}(x,y) = e^{-4y},\ \ 0 < x < 4,\ \ 0 < y$.

a) 

$X \sim \text{Beta}(3,4)$
$Y \sim \text{Beta}(6,1)$

Bonus: $3360 = \binom{8}{3,1,2,1,1}$

b) Rewriting as $\lambda e^{-\lambda x}\cdot\left(e^{-\lambda(y-x)}\cdot\frac{\lambda(y-x)}{1!}\right)\cdot\lambda$,



$X \sim \text{Exp}(\lambda)$, $Y \sim \text{Gamma}(3,\lambda)$ using Poisson Arrival Process.

c) $X \sim \text{Unif}(0,4)$, $Y \sim \text{Exp}(4)$. $f_{X,Y}(x,y) = \underbrace{\frac{1}{4}}_{f_X}\cdot\underbrace{4e^{-4y}}_{f_Y}$

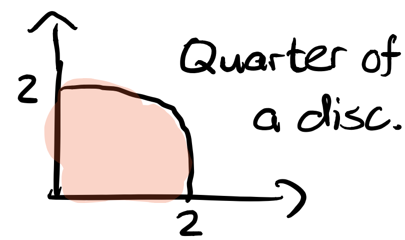
Problem 2

Suppose X, Y follow the standard bivariate normal distribution with correlation ρ. Find the joint density of X and Y. As a reminder, $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is the standard normal PDF.
(Note we have worked a lot with these two variables, but we have not yet derived this density or used it directly!)

$$\mathbb{P}(X \in dx, Y \in dy) = \mathbb{P}(X \in dx, \rho X + \sqrt{1-\rho^2} Z \in dy)$$

$$X = x, Y = y \Rightarrow \frac{y - \rho x}{\sqrt{1-\rho^2}} = Z \qquad = \phi(x)\, \phi\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right).$$

Problem 3



Let (X, Y) represent a point chosen uniformly at random from the region $\{(x, y) : x > 0,\ y > 0,\ x^2 + y^2 < 4\}$. Let R represent the distance from the origin to the random point (X, Y), i.e. $R = \sqrt{X^2 + Y^2}$. Find:

a. $f_{X,Y}(x, y)$;

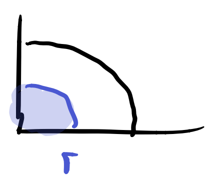
b. $f_R(r)$;

c. $P(cX > Y)$, for some $c > 0$.

a) Area of region $= \frac{\pi (2)^2}{4} = \pi$.

$\Rightarrow f_{X,Y}(x,y) = \frac{1}{\pi}$ for (x,y) in region.

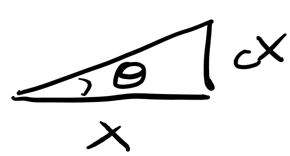
b) Use CDF. $F_R(r) = \mathbb{P}(R < r) = \frac{\left(\frac{\pi r^2}{4}\right)}{\pi} = \frac{r^2}{4}$



$\Rightarrow f_R(r) = \frac{r}{2}, \ 0 < r < 2.$

c)

Reduces to finding θ.



$\theta = \arctan\left(\frac{cX}{X}\right) = \arctan(c)$

Prepared by Brian Thorsen

$$\mathbb{P}(cX > Y) = \frac{\arctan(c)}{(\pi/2)}.$$