Stat 134 lec 16

Warmup 10:00–10:10

A tub of blueberry muffin batter has $\lambda = 2 \text{ bb}/\text{in}^3$ intensity of bb.

bb muffin 1 is 3 in^3

bb muffin 2 is 4 in^3

a) on average how many bb is in muffin 1?

$\mu_1 = \lambda \cdot 3 = \boxed{6 \text{ bb}}$

b) Find P(5 bb in each muffin)

$X_1 \sim Pois(6)$

$X_2 \sim Pois(8)$

$P(X_1=5, X_2=5) = \boxed{\frac{e^{-6}6^5}{5!} \cdot \frac{e^{-8}8^5}{5!}}$

c) Find P(10 bb total in both muffins together).

$X_1 + X_2 \sim Pois(14)$

$P(X_1+X_2=10) = \boxed{\frac{e^{-14}14^{10}}{10!}}$

Announcements

For Monday's review, write down questions in discussion board on b-course by Sunday 8pm.

Last time

sec 3.5 Poisson distribution

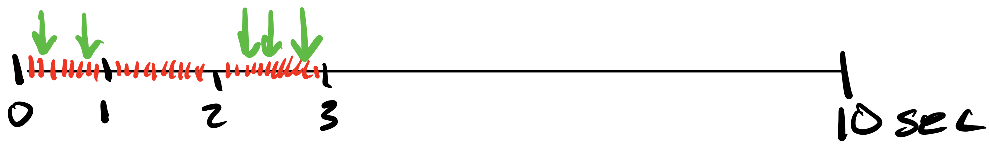
$$X \sim Pois(\mu)$$

$$P(X=k) = \frac{e^{-\mu}\mu^{k}}{k!}$$

$$E(X) = Var(X) = \mu.$$

Poisson Process or Poisson Random Scatter (PRS):

ex radioactive decay of Americium 241 in 10 seconds



Assumptions

1. no two particles arrive at the same time. (this allows us to divide 10 sec into n small time intervals each with at most one arrival.
2. X is a sum of n ← large iid Bernoulli(p) trials.
   ↖ small

$\mu = np$ is avg # of arrivals in 10 sec.
$\lambda = \mu/10$ is the arrival rate per second.

X = # arrivals in 10 seconds.
Suppose $\lambda = 4$ arrivals/sec
then $\mu = \lambda \cdot 10 = 40 \Rightarrow X \sim \text{Pois}(40)$

Americium has a long half life.

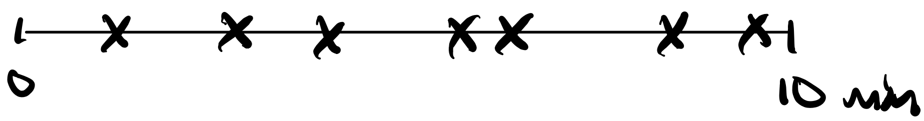
Y = # arrivals in 12070 sec.
$Y \sim \text{Pois}(\lambda \cdot 12070)$
4

<u>Today</u> ① Finish section 3.5 Poisson Thinning
② midterm review

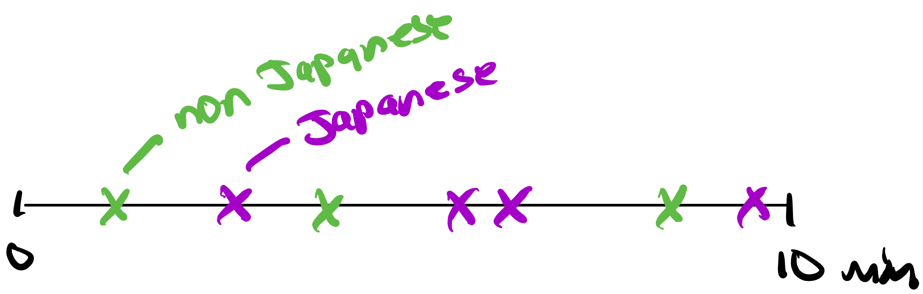
Sec 3.5 Poisson thinning

Cars arrive at a toll booth according to a Poisson process at a rate $\lambda = 3$ arrivals/min

X = # cars arriving at a toll booth in 10 min. $X \sim \text{Pois}(\underbrace{\lambda \cdot 10}_{30})$



Of cars arriving, it is known, over the long term, that 60% are Japanese imports.



Call Japanese cars a success and non Japanese a failure.

$$\text{\# cars} \sim \text{Pois}(\lambda \cdot 10) = \text{Pois}(30)$$

$$\text{\# Japanese imports} \sim \text{Pois}(p\lambda \cdot 10) = \text{Pois}(18)$$

$$\text{\# non Japanese} \sim \text{Pois}(q\lambda \cdot 10) = \text{Pois}(12)$$

Ex: What is the probability that in a given 10 min interval, 15 cars arrive at the booth and 10 are Japanese imports?

$X = \#$ cars in 10 min $\sim \text{Pois}(30)$

indep

$J = \#$ Japanese cars in 10 min $\sim \text{Pois}(18)$

$nJ = \#$ nonJapanese $\sim \text{Pois}(12)$

$$P(X=15, J=10) = P(nJ=5, J=10)$$

$$= P(nJ=5)P(J=10)$$

$$= \boxed{\frac{e^{-12}\,12^{5}}{5!} \cdot \frac{e^{-18}\,18^{10}}{10!}}$$

② midterm review

which distributions are (approximately) a sum of a fixed number of independent Bernoulli trials?

Discrete

name and range	$P(k) = P(X = k)$ for $k \in$ range	mean	variance
✓ uniform on $\{a, a+1, \dots, b\}$ $\{1, 2, \dots, n\}$	$\frac{1}{b-a+1}$	$\frac{a+b}{2}$ $\frac{n+1}{2}$	$\frac{(b-a+1)^2-1}{12}$ $\frac{n^2-1}{12}$
✓ Bernoulli (p) on $\{0, 1\}$	$P(1) = p$; $P(0) = 1 - p$	p	$p(1-p)$
✓ binomial (n, p) on $\{0, 1, \dots, n\}$	$\binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$
✓ Poisson (μ) on $\{0, 1, 2, \dots\}$	$\frac{e^{-\mu}\mu^k}{k!}$	μ	μ
✓ hypergeometric (n, N, G) $(N\ G\ n)$ on $\{0, \dots, n\}$ if $n \ll N$	$\frac{\binom{G}{k}\binom{N-G}{n-k}}{\binom{N}{n}}$	$\frac{nG}{N}$	$n\left(\frac{G}{N}\right)\left(\frac{N-G}{N}\right)\left(\frac{N-n}{N-1}\right)$
geometric (p) on $\{1, 2, 3 \dots\}$	$(1-p)^{k-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
geometric (p) on $\{0, 1, 2 \dots\}$	$(1-p)^k p$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$
negative binomial (r, p) on $\{0, 1, 2, \dots\}$	$\binom{k+r-1}{r-1} p^r (1-p)^k$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$
✓ normal	$\phi(x)$	μ	σ^2

CLT ⇒ normal is sum of iid Bernoulli (p) trials

DeMorgan's rule: $(A \cap B)^c = A^c \cup B^c$

$\Rightarrow A \cap B = (A^c \cup B^c)^c$

so $\boxed{P(A \cap B) = 1 - P(A^c \cup B^c)}$

Inclusion exclusion formula:

Let A_1, A_2, A_3 be dependent RVs with $P(A_i) = .9$ for $i = 1, 2, 3$.

Find a lower bound for $P\left(\bigcap_{i=1}^{3} A_i\right)$

$P\left(\bigcap_{i=1}^{3} A_i\right) = 1 - P\left(\bigcup_{i=1}^{3} A_i^c\right)$ by De Morgan's rule.

$$P\left(\bigcup_{i=1}^{3} A_i^c\right) = P(A_1^c) + P(A_2^c) + P(A_3^c) - P(A_1^c A_2^c) - P(A_2^c A_3^c) - P(A_1^c A_3^c) + P(A_1^c A_2^c A_3^c)$$

$$\Rightarrow P\left(\bigcup_{i=1}^{3} A_i^c\right) \leq P(A_1^c) + P(A_2^c) + P(A_3^c) = 3(.1) = .3$$

$$P\left(\bigcap_{i=1}^{3} A_i\right) \geq 1 - .3 = .7$$

ex Conditional distribution, Poisson

8. Let X_1 and X_2 be independent random variables such that for $i = 1, 2$, the distribution of X_i is Poisson (μ_i). Let m be a fixed positive integer. Find the distribution of X_1 given that $X_1 + X_2 = m$. Recognize this distribution as one of the famous ones, and provide its name and parameters.

$$\left.\begin{array}{l} X_1 \sim \text{Pois}(\mu_1) \\ X_2 \sim \text{Pois}(\mu_2) \end{array}\right\} \text{indep} \qquad P(X_1 = k) = \frac{e^{-\mu_1}\mu_1^k}{k!}$$

$X_1 \mid X_1 + X_2 = m$ takes value $0, 1, 2, \dots, m$

$$P(X_1 = k \mid X_1 + X_2 = m) = \frac{P(X_1 = k, X_2 = m-k)}{P(X_1 + X_2 = m)}$$

$$= \frac{P(X_1 = k)\, P(X_2 = m-k)}{P(X_1 + X_2 = m)}$$

$$= \frac{\frac{\cancel{e^{-\mu_1}}\mu_1^k}{k!} \cdot \frac{\cancel{e^{-\mu_2}}\mu_2^{m-k}}{(m-k)!}}{\frac{\cancel{e^{-(\mu_1+\mu_2)}}(\mu_1+\mu_2)^m}{m!}}$$

$$= \binom{m}{k}\left(\frac{\mu_1}{\mu_1+\mu_2}\right)^k \left(\frac{\mu_2}{\mu_1+\mu_2}\right)^{m-k}$$

$$\Rightarrow X_1 \mid X_1 + X_2 = m \sim \text{Bin}\left(m, \frac{\mu_1}{\mu_1+\mu_2}\right)$$

Problem 4 (conditional probability)

Two jars each contains r red marbles and b blue marbles. A marble is chosen at random from the first jar and placed in the second jar. A marble is then randomly chosen from the second jar. Find the probability this marble is red.

X = the first marble color

Y = the second marble color.

R = red

B = blue

$$\frac{r+1}{r+b+1} \cdot \frac{r}{r+b}$$

$$P(Y=R) = \underbrace{P(Y=R \mid X=R)P(X=R)}_{=\frac{r+1}{r+b+1}\cdot\frac{r}{r+b}}$$

$$+ \underbrace{P(Y=R \mid X=B)P(X=B)}_{=\frac{r}{r+b+1}\cdot\frac{b}{r+b}}$$

expectation question

An urn contains 90 marbles, of which there are 20 greens, 20 blacks and 50 red marbles. Tom draw marble without replacement until the 6^{th} green marble. Let $X = \#$ of marbles drawn. Example: **GGG**BRB**GG**BR**G** with $x = 11$. Find $\mathbb{E}[X]$.

Hint First find the expected number of marbles until the 1^{st} green marble.

What is the min and max of X? 1, 71

$$X = I_1 + \cdots + I_{70} + 1$$

where $I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ nongreen is before } 1^{st} \text{ green} \\ 0 & \text{else} \end{cases}$ $p = 1/21$

2^{nd} nongreen must go here

$\_\, G_1 \,\_\, G_2 \,\_\, \cdots \,\_\, G_{20} \,\_$

$p = 1/21$ since there are 21 slots the 2^{nd} nongreen can go.

$$\Rightarrow E(X) = 70 \cdot (1/21) + 1$$

To solve original problem by symmetry you expect # nongreen marbles between first and second green also to be $70 \cdot (1/21) + 1$. Similar between 2^{nd} and 3^{rd} green etc

Hence answer is $\boxed{6 \cdot \left[70 (1/21) + 1\right]}$