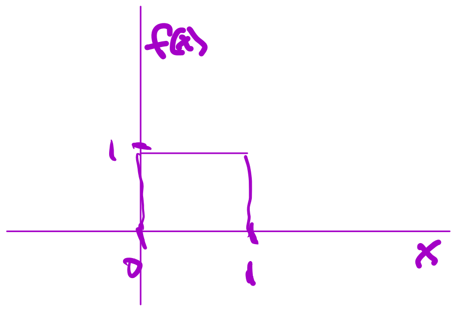
Stat 134 lec 19 (no lec 18)

warmup 10:00–10:10

Let $X \sim Unif(0,1)$ be the standard uniform distribution with histogram (density)

$$f(x) = \begin{cases} 1 & \text{if } 0 < x \le 1 \\ 0 & \text{else} \end{cases}$$

Picture



Define

$$E(X) = \int_{x=-\infty}^{\infty} x f(x)\,dx$$

Find $E(X)$, $E(X^2)$, and $Var(X)$.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \underbrace{\int_{-\infty}^{0} x \cdot 0\,dx}_{=0} + \underbrace{\int_{0}^{1} x \cdot 1\,dx}_{= \frac{x^2}{2}\Big|_0^1 = 1/2} + \underbrace{\int_{1}^{\infty} x \cdot 0\,dx}_{=0} = \boxed{1/2}$$

$$E(X^2) = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \boxed{1/3}$$

$$Var(X) = E(X^2) - (E(X))^2 = 1/3 - (1/2)^2 = \boxed{1/12}$$

Last time

Congratulations on finishing midterm 1 !

today

Sec 4.1 Continuous Distributions

① probability density
② expectation and variance.
③ change of scale

① <u>sec 4.1</u> Probability density.

let X be a continuous RV

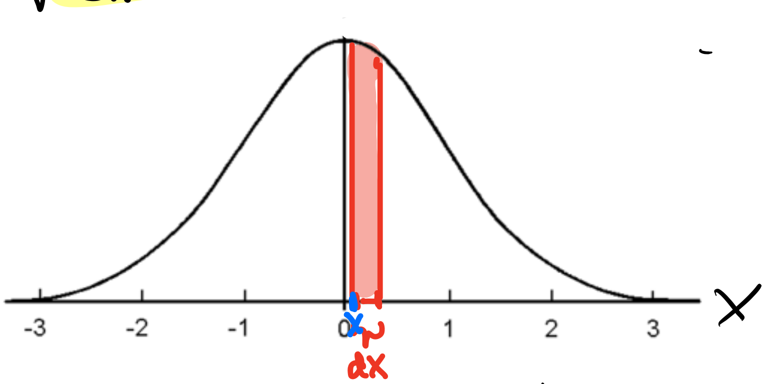
The <u>probability density</u> (histogram) of X is described by a <u>prob densisity function</u>

$$f(x) \geq 0 \text{ for } x \in X$$

and $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

ex The <u>standard normal distribution</u>

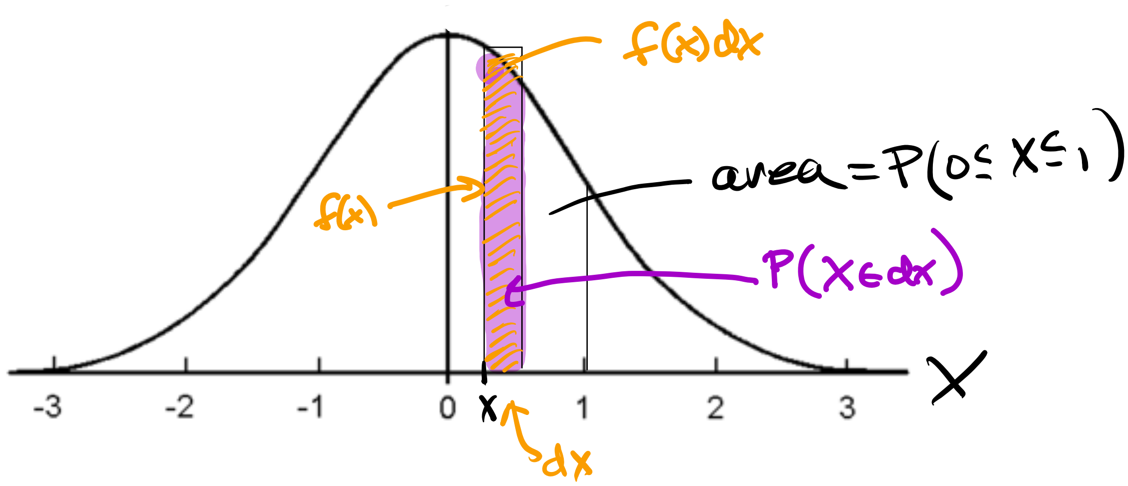
$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$$



Consider a point picked uniformly at random from the area under the curve above the x axis.

The probability of getting an x coordinate in a small neighborhood of x is written $P(X \in dx)$.

This is the area under the curve above dx divided by the total area (which is 1 here).



we see from the rectangle in the picture,

$P(X\in dx) \approx f(x)dx$ (notice purple and orange area not same)

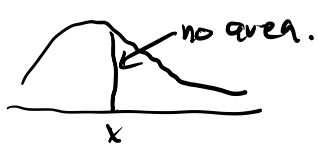
here dx = tiny interval around x and also the length of the interval

For all a, b $P(a\le X\le b) = \int_a^b P(X\in dx) \approx \int_a^b f(x)dx$

Note f(x) is not a probability.
f(x)dx is a probability.

$f(x) \approx \frac{P(X\in dx)}{dx}$ — if dx is in centimeters then units of f(x) is cm^{-1}

$P(X=x)=0$

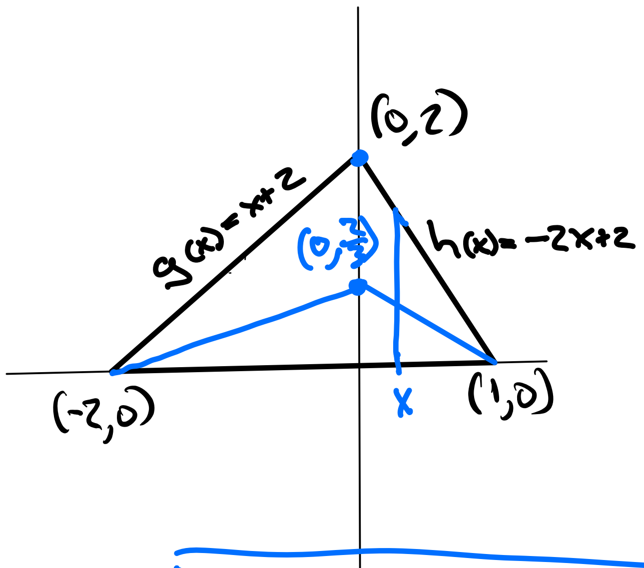


Hence $P(a\le X\le b) = P(a<X<b)$
(we don't have to worry about endpts).

ex 4.1.12 b

this is what I will call a "nonunit density." It is like a density but doesn't have area 1.

Consider a point picked uniformly at random from the area inside the following triangle

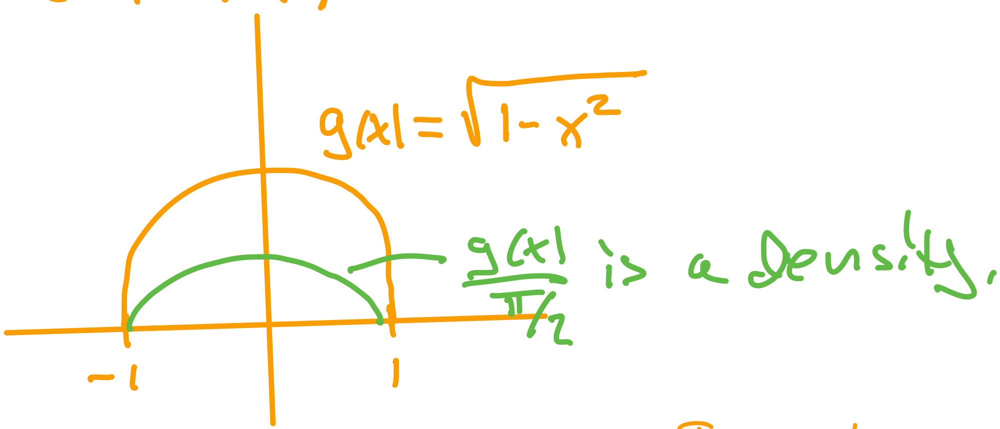


Find the density function of the x-coordinate $f(x)$

We just need to rescale this by dividing it by the total area 3.

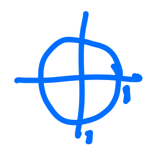
$$\boxed{f(x)=\begin{cases}\frac{x+2}{3} & -2\leq x\leq 0\\ \frac{-2x+2}{3} & 0\leq x\leq 1\\ 0 & \text{else}\end{cases}}$$

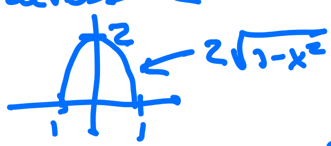
Note there is nothing special about the shape being a triangle, It could be a half circle with radius 1 for example,



Here the area is $\frac{\pi}{2}$, To make g into a density divide it by $\pi/2$

$$f(x)=\frac{\sqrt{1-x^2}}{\pi/2}$$

Suppose the shape was a full circle radius 1, 

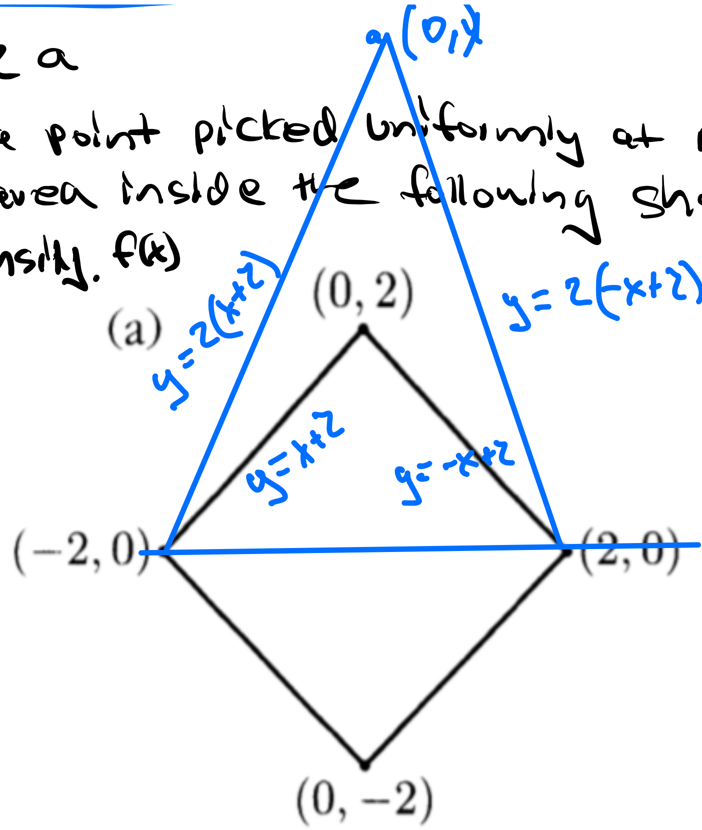
Now part of the shape is under the x axis, If you flip the bottom semicircle across the x axis and add it to the top you get a shape 

that is easier to think about, This is a nonunit density so to find the density of X divide $2\sqrt{1-x^2}$ by the total area π,

$$\boxed{f(x)=\begin{cases}\frac{2\sqrt{1-x^2}}{\pi} & -1\leq x\leq 1\\ 0 & \text{else}\end{cases}}$$

ex 4.1.12 a

Consider a point picked uniformly at random from the area inside the following shape. Find the density. f(x)



$$f(x) = \begin{cases} \frac{2(x+2)}{8} & -2 \le x \le 0 \\ \frac{2(-x+2)}{8} & 0 \le x \le 2 \\ 0 & \text{else} \end{cases}$$

2

Expectation and Variance

For discrete,

$$E(g(X)) = \sum_{x \in X} g(x) P(X=x)$$

For continuous,

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) P(X \in dx) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$Var(X) = E(X^2) - E(X)^2$$

See Wavumbo for example.

(3)

Change of scale

To calculate $E(X)$, $Var(X)$, $P(X \in dx)$ we sometimes make a linear change of scale

$Y = c + bX$ where c, b are constants

Y hopefully has a simpler density function.

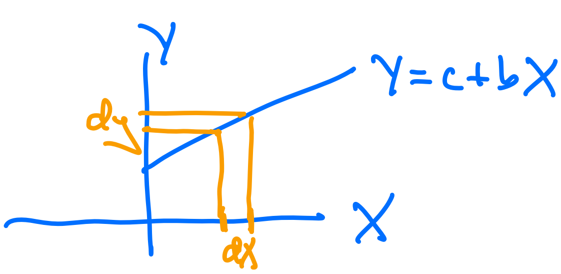
We can recover $E(X)$, $Var(X)$, $P(X \in dx)$ from $E(Y)$, $Var(Y)$, $P(Y \in dY)$.

if $Y = c + bX$ then $X = \frac{Y - c}{b} = \frac{1}{b}Y - \frac{c}{b}$

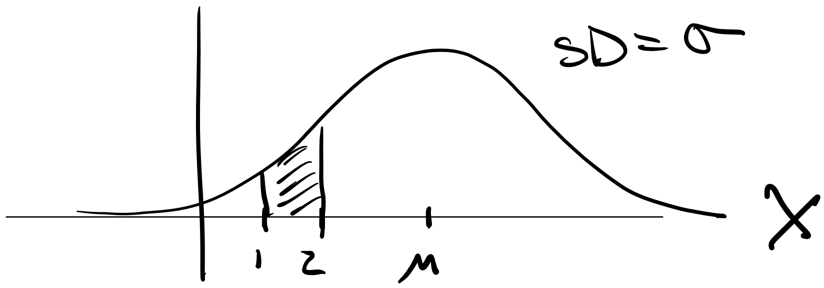
and $\boxed{E(X) = \frac{1}{b}E(Y) - \frac{c}{b}}$

$\boxed{Var(X) = \left(\frac{1}{b}\right)^2 Var(Y)}$

also $P(X \in dx) = P(Y \in dy)$ since the event $X \in dx$ is equivalent to $Y \in dY$



ex Let $X \sim N(\mu, \sigma^2)$
Find $P(1 < X < 2)$



We make a change of scale to the standard normal for which we know how to find probabilities.

let $Z = \frac{X-\mu}{\sigma} = \underbrace{\frac{-\mu}{\sigma}}_{c} + \underbrace{\frac{1}{\sigma}}_{b} X$ change of scale of X

$$P(1 < X < 2) = P\left(\frac{1-\mu}{\sigma} < Z < \frac{2-\mu}{\sigma}\right)$$
$$= \Phi\left(\frac{2-\mu}{\sigma}\right) - \Phi\left(\frac{1-\mu}{\sigma}\right)$$

In this example we used a change of scale to find $P(1 < X < 2)$.