Stat 134 lec 35

warmup: 10:00-10:10

$$P(Y\in dy)=\int_x P(Y\in dy|X=x)f_X(x)dx$$

$X\sim \text{Unif}(0,1)$

$I_1|X=x,\ I_2|X=x \overset{iid}{\sim} \text{Ber}(x)$

Find $P(I_1=1) = \int_{x=0}^{x=1} \overset{\overset{x}{\parallel}}{P(I_1=1|X=x)}\cdot \overset{\overset{1}{\parallel}}{f_X(x)}dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \boxed{1/2}$

Similarly $P(I_2=1)=1/2$

b) Find $P(I_2=1|I_1=1) = \dfrac{P(I_2=1,I_1=1) \leftarrow 1/3}{P(I_1=1) \leftarrow 1/2}$

$P(I_2=1,I_1=1) = \int_{x=0}^{x=1} P(I_2=1,I_1=1|X=x)\cdot f_X(x)dx$

rule of average conditional prob

$$= \int_{x=0}^{x=1} \overset{\overset{x}{\parallel}}{P(I_2=1|X=x)}\overset{\overset{x}{\parallel}}{P(I_1=1|X=x)}\overset{\overset{1}{\parallel}}{f_X(x)}dx$$

$$= \int_{x=0}^{x=1} x^2 dx = \frac{x^3}{3}\Big|_0^1 = \boxed{1/3}$$

$$P(I_2=1|I_1=1) = \frac{1/3}{1/2} = \boxed{2/3}$$

Last time

Sec 6.2

Properties

① $E(Y) = E(E(Y|X))$ iterated expectation

② $E(aY+b|X) = aE(Y|X)+b$

③ $E(Y+Z|X) = E(Y|X)+E(Z|X)$

④ $E(g(X)|X) = g(X)$

⑤ $E(g(X)Y|X) = g(X)E(Y|X)$

⑥ $Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$ total variance decomposition (sec 6.2.18)

$Var(Y|X)$ is called the conditional variance.

ex $X \sim Unif(0,1)$ $\qquad Var(X) = 1/12,\ E(X) = 1/2,\ E(X^2) = 1/3$

$I \mid X=x \sim Ber(x)$

What is $Var(I)$?

$$Var(I|X) = X(1-X)$$

$$E(I|X) = 1 \cdot X + 0(1-X) = X$$

$$Var(I) = E(Var(I|X)) + Var(E(I|X))$$

$$= E(X(1-X)) + Var(X) = 1/6 + 1/12 = 3/12 = \boxed{1/4}$$

$$= ?\ \ 1/2 - 1/3 \qquad\qquad 1/12$$

Sec 6.3 Conditional densities.

Conditional Prob mass function: $\boxed{P_{Y|X=x}(y) = \frac{P(x,y)}{P(x)}}$

(discrete X, Y)

Conditional density: $\boxed{f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)}}$

(continuous X, Y)

Rule of average conditional probabilities (discrete case)

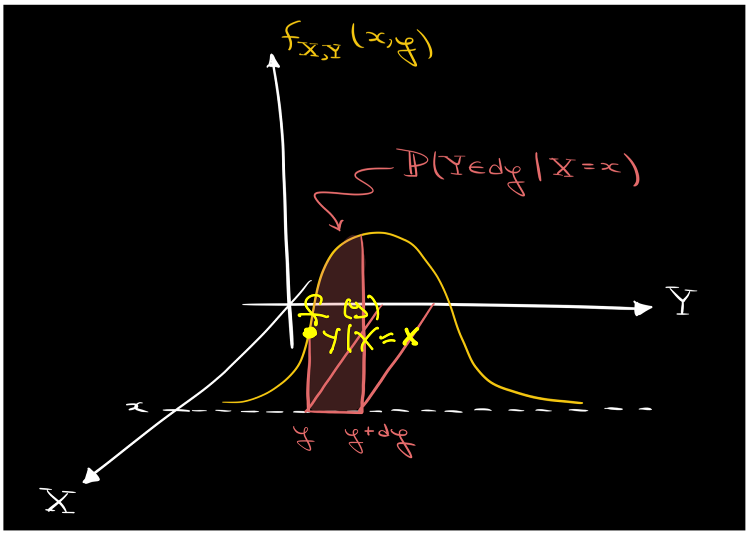
Let X and Y be discrete RV w/ joint distribution $P(X=x, Y=y)$

$$P(Y=y) = \sum_x P(Y=y|X=x)P(X=x)$$

Rule of average conditional probabilities (Continuous case)

$$P(Y\in dy) = \int_{x\in X} P(Y\in dy|X=x)\, f_X(x)dx$$

$$= \int_{x\in X} f_{Y|X=x}(y)dy\, f_X(x)dx$$



The multiplication rule is

$X \sim \text{Gamma}(r, \lambda)$

$f_X(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, \quad x > 0$

$f(x,y) = f_{Y|X=x}(y) f_X(x)$

ex

let $X \sim \text{Gamma}(2, \lambda)$

$Y|X=x \sim \text{Unif}(0, x)$

a) Find $f_{Y|X=x}(y) = \begin{cases} \frac{1}{x} & \text{for } 0<y<x \\ 0 & \text{else} \end{cases}$

b) Find $f(x,y) = \underset{f_{Y|X=x}(y)}{\underset{\shortparallel}{\frac{1}{x}}} \cdot \underset{f_X(x)}{\underset{\shortparallel}{\lambda^2 x e^{-\lambda x}}} = \boxed{\lambda^2 e^{-\lambda x}, \quad 0<x<\infty}$

Today Sec 6.3

① Bayesian statistics

Sec 6.3

① Bayesian Statistics

In **frequentist** statistics we interpret probability as a long run average constant known only to Tyche, the goddess of fortune.

In **Bayesian** statistics we interpret probability as a RV

ex When probability a coin lands head is a RV X rather than an unknown constant we are doing Bayesian statistics,

i.e

$$X \sim \text{Unif}(0,1)$$

$$I_1|X=x,\ I_2|X=x \overset{iid}{\sim} \text{Ber}(x)$$

CAUTION X is continuous and I_1 is discrete

We write $P(I_1|X=x)$ for conditional probability mass function (pmf) of I_1 and $f_{X|I_1=1}(x)$ for the conditional density of X

$$P(I_1=1, X=x) \overset{\text{multiplication rule}}{=} P(I_1=1|X=x)\cdot f_X(x)$$

$$\|$$

$$P(X=x, I_1=1) \overset{\text{multiplication rule}}{=} f_{X|I_1=1}(x)\, P(I_1=1)$$

$$\overset{\text{posterior}}{f_{X|I_1=1}(x)} = \frac{\overset{\text{likelihood}}{P(I_1=1|X=x)}\cdot \overset{\text{prior}}{f_X(x)}}{\underbrace{P(I_1=1)}_{\text{constant,}}}$$

Posterior $\propto$ likelihood · Prior

ex Find $f_{X|I_1=1}(x) = \dfrac{x \cdot 1}{1/2} = 2x$

Review Beta Distribution

variable part,

$X \sim \text{Beta}(r,s)$

$$f_X(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}, \quad 0<x<1$$

where $r \in \mathbb{Z}^+ \Rightarrow \Gamma(r) = (r-1)!$

ex If $0<x<1$,

$f_X(x) \propto 1 \Rightarrow X \sim \text{Beta}(1,1)$

$f_X(x) \propto x \Rightarrow X \sim \text{Beta}(2,1)$

$f_X(x) \propto x(1-x) \Rightarrow X \sim \text{Beta}(2,2)$

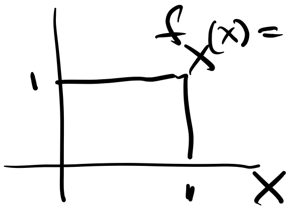
ex $X \sim \text{Unif}(0,1)$

$I_1|X=x, I_2|X=x \overset{iid}{\sim} \text{Ber}(x)$

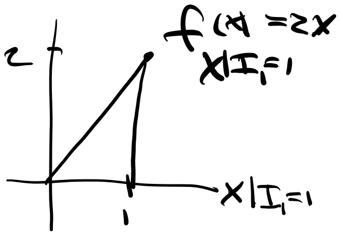
Prior density $f_X(x) = 1 \Rightarrow X \sim \text{Unif}(0,1) = \text{Beta}(1,$

Posterior density $f_{X|I_1=1}(x) = 2x \Rightarrow X|I_1=1 \sim \text{Beta}(2,1)$

Prior $X \sim \text{Unif}(0,1)$



Posterior



ex Let A be an event and

$X \sim \text{Unif}(0,1)$

Suppose $P(A \mid X=x) = x$

Find $f_{X \mid A^c}(x)$

$X \sim \text{Beta}(r,s)$

$$f_X(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \underbrace{x^{r-1}(1-x)^{s-1}}_{\text{variable part.}}, \quad 0<x<1$$

$$f_{X \mid A^c} \propto \text{likelihood} \cdot \text{prior}$$

$$= P(A^c \mid X=x) \cdot 1$$

$$= 1-x$$

$$X \mid A^c \sim \boxed{\text{Beta}(1,2)}$$

Stat 134

1. Let A, B and C be events and let X be a random variable uniformly distributed on (0,1). Suppose conditional on X=x, that A, B, and C are independent each with probability x. The conditional density of X given that A and B occurs and C doesn't is: $\underline{\text{i.e}}$ $X | ABC^c \sim ?$

   a $Beta(2, 2)$

   b $Beta(3, 2)$

   c $Beta(2, 3)$

   d none of the above