Stat 134 lec 37

warmup 10:00-10:10

1. Let A, B and C be events and let X be a random variable uniformly distributed on (0,1). Suppose conditional on X=x, that A, B, and C are independent each with probability x. The conditional density of X given that A and B occurs and C doesn't is: i.e $X|ABC^c \sim$?

   a $Beta(2,2)$

   [✓ **b** $Beta(3,2)$]

   c $Beta(2,3)$

   d none of the above

$X \sim Beta(r,s)$

$f_X(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}$, $0<x<1$

variable part

$$f_{X|ABC^c} \propto \text{likelihood} \cdot \text{prior}$$

$$= \underbrace{P(ABC^c|X=x)}_{\parallel} \overset{\parallel}{f_X(x)}$$

$$\underset{\overset{\parallel}{x}}{P(A|X=x)}\underset{x}{P(B|X=x)}\Big(1-\underset{\overset{\parallel}{(1-x)}}{P(C|X=x)}\Big)$$

$$= x^2(1-x)$$

Last time

Rule of average conditional probabilities (mixed case : Y discrete, X cont.)

$$P(Y=y) = \int_{x\in X} P(Y=y|X=x) f_X(x)dx$$

ex Suppose $Y|X=x \sim \text{Pois}(x)$ where $X\sim \text{Exp}(\lambda)$.

Show $Y \sim \text{Geom}\left(\overset{p}{\boxed{\frac{\lambda}{\lambda+1}}}\right)$ on $0,1,2,\ldots$

$P(Y=y) = q^y p \quad y=0,1,2,\ldots$

$$P(Y=y) = \int_{x=0}^{x=\infty} P(Y=y|X=x) f_X(x)dx$$

$$= \int_{x=0}^{x=\infty} \frac{e^{-x}x^y}{y!}\lambda e^{-\lambda x}dx$$

$$= \frac{\lambda}{y!}\int_0^\infty \underbrace{x^y \overset{= e^{-x(\lambda+1)}}{e^{-x}\cdot e^{-\lambda x}}}_{\text{var part of Gamma}(y+1,\ \lambda+1)} dx$$

$$\| \quad \frac{1}{\text{const part}} = \frac{1}{\frac{(\lambda+1)^{y+1}}{\Gamma(y+1)}} = \frac{y!}{(\lambda+1)^{y+1}}$$

$X\sim \text{Gamma}(r,\alpha)$

$\Rightarrow f_X(x) = \frac{\alpha^r}{\Gamma(r)} x^{r-1} e^{-x\alpha}$

$$= \frac{\lambda}{\cancel{y!}}\cdot\frac{\cancel{y!}}{(\lambda+1)^{y+1}} = \left(\frac{1}{\lambda+1}\right)^y \cdot \Big(\underbrace{\frac{\lambda}{\lambda+1}}_{p}\Big)$$

$$\frac{1}{\lambda+1} \leftarrow \Big(\underbrace{1-\left(\frac{\lambda}{\lambda+1}\right)}_{q}\Big)^y$$

Bayesian statistics

Posterior $\propto$ likelihood $\cdot$ Prior

Today

① **Sec 6.3** conjugate pairs

② **Sec 6.4** covariance and the variance of sum

① Sec 6.3 conjugate pairs

The posterior can be difficult to calculate except when the prior and likelihood are conjugate pairs:

ex prior $X \sim \text{beta}(r, s)$

likelihood $Y \sim \text{Bin}(n, X)$

$$\text{Posterior} \propto \text{likelihood} \cdot \text{Prior}$$

$$f_{X|Y=j}(x) \propto P(Y=j \mid X=x)\, f_X(x)$$

similar

$$\underbrace{\overbrace{x^{j}(1-x)^{n-j}} \cdot \overbrace{x^{r-1}(1-x)^{s-1}}}_{=}$$

$$x^{j+r-1}(1-x)^{n-j+s-1}$$

$$\Rightarrow X \mid Y=j \sim \text{Beta}(j+r,\, n-j+s)$$

Defn (conjugate pairs)

The prior and likelihood are conjugate pairs when the prior and posterior belong to the same distribution family.

eg Let $\overbrace{Y|X=x \sim \text{Pois}(x)}^{\text{likelihood}}$ where $\overbrace{X \sim \text{Exp}(\lambda)}^{\text{prior}}$

What distribution is $\overbrace{X|Y=y}^{\text{posterior}}$?

Posterior ∝ likelihood · prior

$$f_{X|Y=y}(x) = \frac{P(Y=y|X=x)f_X(x)}{P(Y=y)}$$

$$f_{X|Y=y}(x) \propto \underbrace{\frac{e^{-x}x^y}{y!}}_{\parallel \atop P(Y=y|X=x)} \cdot \lambda e^{-\lambda x} = \frac{\lambda}{y!} \underbrace{x^y e^{-x(\lambda+1)}}_{}$$

ker part of Gamma(y+1, λ+1)

$\Rightarrow X|Y=y \sim \text{Gamma}(y+1, \lambda+1)$

$\Rightarrow$ Prior = Gamma, and likelihood = Poisson are conjugate pairs.

ex Suppose $\Theta \sim \text{Gamma}(r, \lambda)$ with r, λ known.

Let $(N_1|\Theta=\theta, N_2|\Theta=\theta, N_3|\Theta=\theta) \overset{iid}{\sim} \text{Poi}(\theta)$.

Find the posterior distribution of Θ.

Gamma: $f_\Theta(\theta) \propto \theta^{r-1} e^{-\lambda\theta}$

Poisson: $P(N=n|\Theta=\theta) \propto e^{-\theta}\theta^{n}$

$$f_{\Theta|N_1=n_1, N_2=n_2, N_3=n_3} \propto \text{likelihood} \cdot \text{prior}$$

$$\propto P(N_1=n_1, N_2=n_3\, N_3=n_3 | \Theta=\theta) \cdot f_\Theta(\theta)$$

$$(e^{-\theta}\theta^{n_1}) \cdot (e^{-\theta}\theta^{n_2})(e^{-\theta}\theta^{n_3}) \cdot \theta^{r-1} e^{-\lambda\theta}$$

$$= \theta^{(n_1+n_2+n_3)+r-1} e^{-(3+\lambda)\theta}$$

$\Rightarrow$ Posterior is Gamma $(n_1+n_2+n_3+r, 3+\lambda)$

$\Rightarrow$ prior = Gamma and likelihood = Poisson is a conjugate pair.

② Sec 6.4 Covariance and variance of a sum

$X, Y, \quad S = X + Y$

mean $\mu_X, \mu_Y, \mu_S = \mu_X + \mu_Y$

$D_S = S - \mu_S$ D_S deviation from mean

$= X + Y - (\mu_X + \mu_Y)$

$= D_X + D_Y$

$E(D_S^2) - (E(D_S))^2 = E(D_S^2)$ Note $E(D_S) = 0$

$$\mathrm{Var}(S) = E\left((D_X + D_Y)^2\right)$$

$$= E(D_X^2 + D_Y^2 + 2D_XD_Y)$$

$$= \underset{\mathrm{Var}(X)}{\underset{\shortparallel}{E(D_X^2)}} + \underset{\mathrm{Var}(Y)}{\underset{\shortparallel}{E(D_Y^2)}} + 2\,\underbrace{E(D_XD_Y)}_{\overset{\shortparallel}{\mathrm{Cov}(X,Y)}}$$

Def^n The covariance of X and Y is

$$\mathrm{Cov}(X,Y) = E\big(\underset{D_X}{\underset{\shortparallel}{(X-\mu_X)}}\underset{D_Y}{\underset{\shortparallel}{(Y-\mu_Y)}}\big)$$

Bilinearity Properties

Proved end of lecture,

ⓐ $\mathrm{Cov}(X+Y, Z) = \mathrm{Cov}(X,Z) + \mathrm{Cov}(Y,Z)$

ⓑ $\mathrm{Cov}(aX, bY) = ab\,\mathrm{Cov}(X,Y)$

More generally

$$\mathrm{Cov}\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \mathrm{Cov}(X_i, Y_j)$$

Proved end of lecture,

Thm $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$

Easy facts

$\mathrm{Cov}(X,X) = E(X^2) - E(X)^2 = \mathrm{Var}(X)$

$\mathrm{Cov}(X,X) = \mathrm{Var}(X)$

$\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$

$\mathrm{Cov}(X,c) = 0$

constant

ex

Simplify

$\mathrm{Cov}(X-5Y,\ 3X+Y-Z+10)$

$= 3\mathrm{Var}(X) + \mathrm{Cov}(X,Y) - \mathrm{Cov}(X,Z) + 0$

$- 15\mathrm{Cov}(X,Y) - 5\mathrm{Var}(Y) + 5\mathrm{Cov}(Y,Z) + 0$

Recall X, Y independent

$\Rightarrow \quad E(XY) = E(X)E(Y)$

$\mathrm{Cov}(X,Y) = 0$ if X, Y independent,

Hence if X, Y indep,

$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\overset{0}{\overset{\shortparallel}{\mathrm{Cov}}}(X,Y)$

$= \mathrm{Var}(X) + \mathrm{Var}(Y)$

Stat 134

Wednesday April 24 2019

1. Consider a Poisson(λ) process. Let $T_r \sim$ gamma(r, λ) be the rth arrival time. Cov(T_1, T_3) equals:

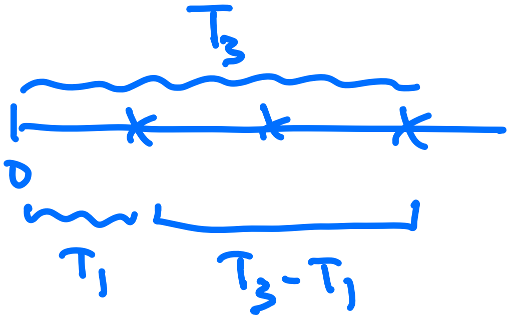
a λ

b λ^2

[✓ **c** $1/\lambda^2$]

d none of the above

Recall $\quad Var(T_r) = \frac{r}{\lambda^2}$



$Cov(T_1, T_3 - T_1) = 0$ since T_1 and $T_3 - T_1$ independent

$\parallel$

$Cov(T_1, T_3) - Var(T_1)$

$Cov(T_1, T_3) = Var(T_1) = \frac{1}{\lambda^2}$

Appendix

Bilinearity Properties

Thm

ⓐ $\mathrm{Cov}(X+Y, Z) = \mathrm{Cov}(X,Z) + \mathrm{Cov}(Y,Z)$

ⓑ $\mathrm{Cov}(aX, bY) = ab\,\mathrm{Cov}(X,Y)$

Pf/

a)

$$\mathrm{Cov}(X+Y,Z) = E((X+Y-\mu_{X+Y})(Z-\mu_Z))$$

$$= E([(X-\mu_X)+(Y-\mu_Y)](Z-\mu_Z))$$

$$= E((X-\mu_X)(Z-\mu_Z) + (Y-\mu_Y)(Z-\mu_Z))$$

$$= E((X-\mu_X)(Z-\mu_Z)) + E((Y-\mu_Y)(Z-\mu_Z))$$

$$= \mathrm{Cov}(X,Z) + \mathrm{Cov}(Y,Z). \quad \square$$

b)

$$\mathrm{Cov}(aX,bY) = E((aX-\mu_{aX})(bY-\mu_{bY}))$$

$$= E((aX-a\mu_X)(bY-b\mu_Y))$$

$$= E(ab(X-\mu_X)(Y-\mu_Y))$$

$$= abE((X-\mu_X)(Y-\mu_Y))$$

$$= ab\,\mathrm{Cov}(X,Y) \quad \square$$

Appendix

Thm $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$

pf/ $\mathrm{Cov}(X,Y) = E(D_x D_y) = E((X-\mu_x)(Y-\mu_y))$

$= E(XY - \mu_x Y - X\mu_y + \mu_x \mu_y)$

$= E(XY) - \mu_x\mu_y - \mu_x\mu_y + \mu_x\mu_y)$

$= E[XY) - E[X]E(Y)$

□