Stat 134 lec 9

Warmup 10:00–10:10

(X_1, X_2) has joint distribution:

		$\frac{1}{6}$	$\frac{5}{6}$
		$X_2=0$	$X_2=1$
$\frac{5}{6}$	$X_1=0$	$\frac{5}{36}$	$\frac{25}{36}$
$\frac{1}{6}$	$X_1=1$	$\frac{1}{36}$	$\frac{5}{36}$

Is X_1, X_2 independent?

chk

$P(0,0) = P(0)P(0)$ ✓

$\overset{\shortparallel}{5/36} \quad \overset{\shortparallel}{5/6} \quad \overset{\shortparallel}{1/6}$

$P(0,1) = P(0)P(1)$ ✓

$P(1,0) = P(1)P(0)$ ✓

$P(1,1) = P(1)P(1)$ ✓

If $P(x,y) \neq P(x)P(y)$ we say x, y are dependent.

<u>Note</u> The entries in the cells are $P(X=x, Y=y)$ not $P(X=x \mid Y=y)$. To find $P(X=x \mid Y=y)$ use Baye's rule

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}.$$

Last time

Sec 3.1 Random Variables

The event $(X=x, Y=y)$ is the intersection of events $X=x$ and $Y=y$. ← sometimes written (x,y)

The probability X and Y satisfies some condition (i.e $P(X+Y=s)$) is the sum of $P(x,y)$ that satisfy that condition.

ex $$P(X+Y=s) = \sum_{(x,y):\,x+y=s} P(x,y) = \sum_{\text{all } x} P(x, s-x)$$ addition rule

Independence of (X,Y,Z) means

$P(X=x, Y=y, Z=z) = P(X=x)P(Y=y)P(Z=z)$ for all $x \in X$, $y \in Y$, $z \in Z$.

Today

1. Sec 3.1 Sums of independent Poissons is Poisson
2. Sec 3.2 Expectation of a RV.

① Sum of independent Poisson is Poisson

informal argument:

$X_1 \sim Bin(1000, \frac{1}{1000})$ $\approx Pois(1)$

$X_2 \sim Bin(2000, \frac{1}{1000})$ $\approx Pois(2)$

indep

$X_1 + X_2 \sim ?$ $Bin(3000, \frac{1}{1000}) \approx Pois(3)$

$X_1 + X_2$ = # heads in 1000+2000 = 3000 $p = \frac{1}{1000}$ coin tosses.

Lets prove this rigorously:

Recall binomial theorem

$$(a+b)^3 = \binom{3}{3} a^3 b^0 + \binom{3}{2} a^2 b^1 + \binom{3}{1} a^1 b^2 + \binom{3}{0} a^0 b^3$$

$$= a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Recall $X \sim Pois(\mu)$

$$P(X=k) = \frac{e^{-\mu} \mu^k}{k!}$$

Claim If $X \sim \text{Pois}(\mu)$ and $Y \sim \text{Pois}(\lambda)$ are independent then

$$S = X + Y \sim \text{Pois}(\mu + \lambda).$$

Pf/

addition rule

$$P(S=s) = P(X=0, Y=s) + P(X=1, Y=s-1) + \dots P(X=s, Y=0)$$

summation notation

$$= \sum_{k=0}^{s} P(X=k, Y=s-k)$$

indep of X, Y

$$= \sum_{k=0}^{s} P(X=k)\,P(Y=s-k)$$

poisson formula

$$= \sum_{k=0}^{s} \frac{e^{-\mu}\mu^{k}}{k!} \cdot \frac{e^{-\lambda}\lambda^{s-k}}{(s-k)!}$$

$\frac{s!}{s!} = 1$

$$= e^{-(\lambda+\mu)} \frac{1}{s!} \sum_{k=0}^{s} \frac{s!}{k!(s-k)!} \mu^{k} \lambda^{s-k}$$

binomial thm

$$= e^{-(\lambda+\mu)} \frac{1}{s!} (\mu+\lambda)^{s}$$

$$\Rightarrow S \sim \text{Pois}(\mu+\lambda).$$

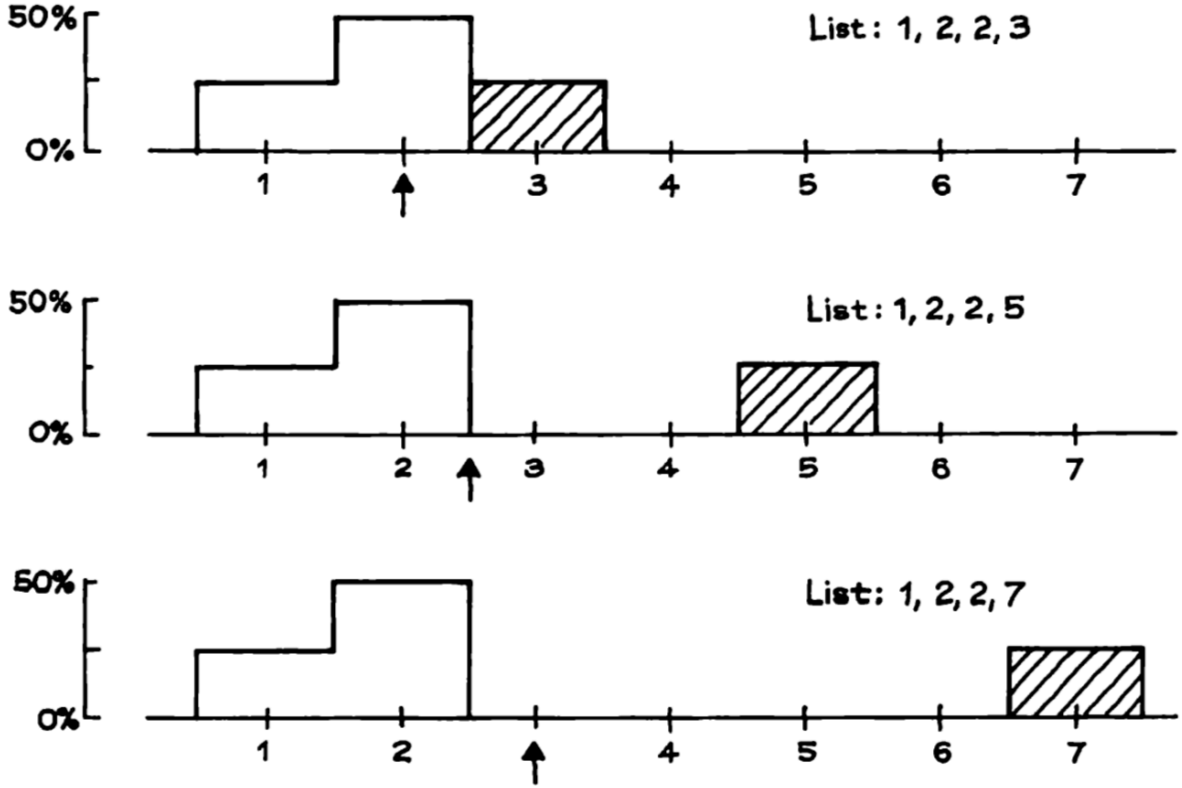
poisson formula.

$\square$

② Sec 3.2 Expectation

$$E(X) = \sum_{x \in X} x \cdot P(X=x)$$

$$E(X) = 1\cdot\tfrac{1}{4} + 2\cdot\tfrac{1}{2} + 3\cdot\tfrac{1}{4} = 2$$



$$E(X) = \sum_{x \in X} x \cdot P(X=x)$$

Properties of Expectation — P167 Pitman

① $E(c) = c$

② $E(X+Y) = E(X) + E(Y)$ (X, Y <u>dont</u> need to be independent)

③ $E(aX+b) = aE(X) + b$

Indicators

An indicator is a RV that has only 2 values 1 (w/ prob p) and 0 (with prob 1−p).

$$I = \begin{cases} 1 & \text{prob } p \\ 0 & \text{prob } 1-p \end{cases}$$

same as a Bernoulli p trial.

$$E(I) = 1 \cdot p + 0 \cdot (1-p) = p$$

RV that are Counts can often be written as a sum of indicators.

ex $X \sim Bin(n,p)$

\# successes in n Bernoulli p trials,

ex X = # heads in n flips of p coin

$$X = I_1 + I_2 + \cdots + I_n$$

where $I_j = \begin{cases} 1 & \text{if } j^{th} \text{ trial success} \\ 0 & \text{else} \end{cases}$ p

$$E(X) = \underbrace{E(I_1)}_{= p} + \cdots + \underbrace{E(I_n)}_{= p} = \boxed{np}$$

indicators are independent since trials are indep.

Ex $X = \#$ aces in a poker hand from a deck of cards

$$X \sim HG(5, 52, 4)$$

a) what are the range of values of X?

0, 1, 2, 3, 4

b) write X as a sum of indicators

$$X = I_1 + I_2 + I_3 + I_4 + I_5$$

$p = 4/52$

c) How is I_2 defined?

$$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ card is ace} \\ 0 & \end{cases}$$

d) Find $E(I_2)$

$$E(I_2) = \frac{4}{52}$$

e) Find $E(X)$

$$X = I_1 + I_2 + \dots + I_5$$

$$E(X) = E(I_1 + \dots + I_5) = \underset{p}{E(I_1)} + \dots + \underset{p}{E(I_5)} = \boxed{5\left(\frac{4}{52}\right)}$$

Note

You may define $I_2 = \begin{cases} 1 & \text{if get 2 aces} \\ 0 & \text{else} \end{cases}$

so

$$X = I_1 + 2I_2 + 3I_3 + 4I_4$$

This is also correct but more complicated.

$$E(I_1) = \frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}} \qquad E(I_3) = \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}$$

$$E(I_2) = \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} \qquad E(I_4) = \frac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$$

so $E(X) = \frac{1}{\binom{52}{5}} \left[\binom{4}{1}\binom{48}{4} + 2 \cdot \binom{4}{2}\binom{48}{3} + 3 \cdot \binom{4}{3}\binom{48}{2} + 4\binom{4}{4}\binom{48}{1} \right]$

$= 5 \cdot \left(\frac{4}{52}\right)$ ← I checked this in R