Stat 134 lec 13

warmup

Let X = number of sixes in 7 tosses of a fair die. $X \sim Bin(7, \frac{1}{6})$

a) write X as a sum of indicators

$X = I_1 + I_2 + \cdots + I_7$ $\quad p = 1/6$

b) Find $Var(X)$

$$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ toss is a 6} \\ 0 & \text{else} \end{cases}$$

$$Var(X) = Var(I_1 + I_2 + \cdots + I_7)$$
$$= \underbrace{Var(I_1)}_{pq} + Var(I_2) + \cdots + Var(I_7)$$
$$= \boxed{7pq}$$

$X \sim Bin(n, p)$

$Var(X) = npq$

If n is large p small and $np \to \mu$ then $X \sim Pois(\mu)$

$Var(X) = npq \to \mu \cdot 1 = \mu$

__Last time__

__Sec 3.3__ $Var(X) = E\left(\left(X - E(X)\right)^2\right)$

or $Var(X) = E(X^2) - \left(E(X)\right)^2$

__ex__ $I = \begin{cases} 1 & \text{with prob } p \\ 0 & \text{with prob } q \end{cases}$

$var(I) = pq$

see p 193 Pitman for the proof.

__Thm__ $Var(X+Y) = Var(X) + Var(Y)$ if X, Y are independent.

__ex__ $X \sim Bin(n, p)$

$var(X) = npq$

$SD(X) = \sqrt{npq}$

Stat 134

1. X is nonnegative random variable with $E(X) = 3$ and $SD(X) = 2$. True, False or Maybe:

$$P(X^2 \geq 40) \leq \frac{1}{3}$$

a True

b False

c Maybe

maybe

b	Both Markov's and Chebyshev's give values bigger than 1/3 when we think about P(X>= sqrt(40))

$$P(X^2 \geq 40) = P(X \geq \sqrt{40}) \leq \frac{3}{\sqrt{40}} > 1/3$$

X nonneg

a	We can solve for E[x^2] and using Markov Inequality, we go 1/4 and it is true for both that it is less than 1/4 and 1/3.

Today

(0) Property of variance

(1) Sec 3.3 Central Limit theorem (CLT)

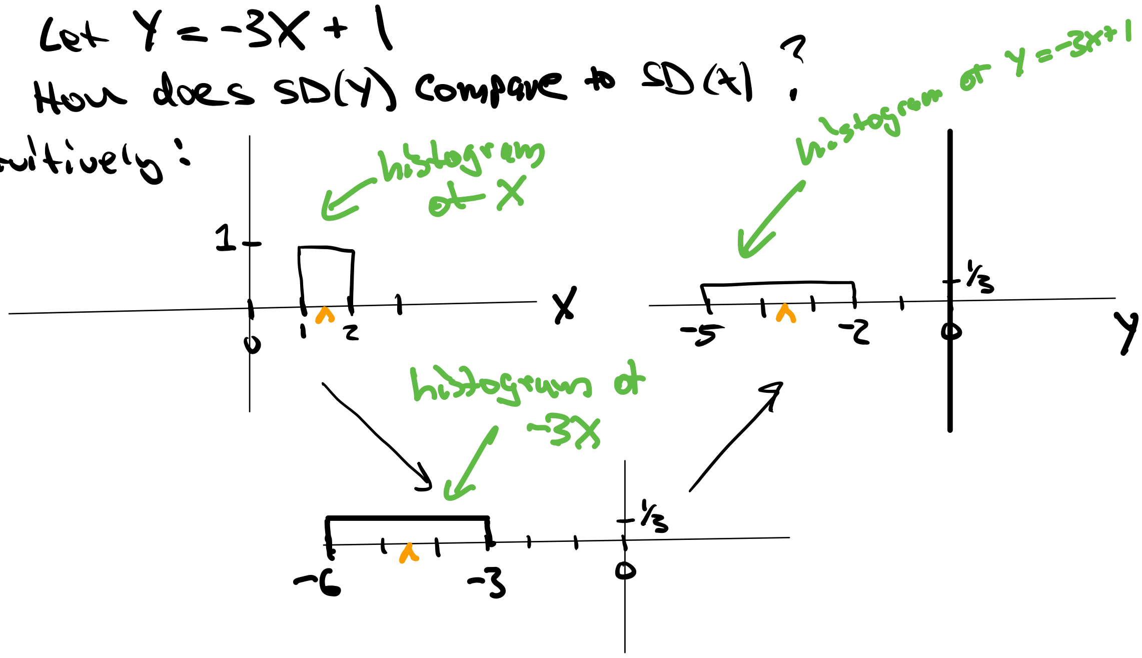
(2) Sec 3.6 (next time sec 3.4) Calculating the variance of a sum of dependent indicators.

⓪ Property of Variance

Let $Y = -3X + 1$

How does $SD(Y)$ compare to $SD(X)$?

intuitively:



$$\boxed{\begin{aligned} SD(aX+b) &= |a|\,SD(X) \\ Var(aX+b) &= a^2\,Var(X) \end{aligned}}$$

Central Limit Thm (CLT)

Let $S_n = X_1 + \cdots + X_n$ where $X_1, \ldots, X_n$ are iid RVs, $E(X) = \mu$, $Var(X) = \sigma^2$.

Then,

$$S_n \approx N(n\mu, n\sigma^2) \text{ for "large" } n.$$

approximately

often ≥ 10

ex

Let $X_1, X_2, \ldots X_{10}$ be i.i.d. Poisson(1).

Let $S_{10} = X_1 + \cdots + X_{10}$

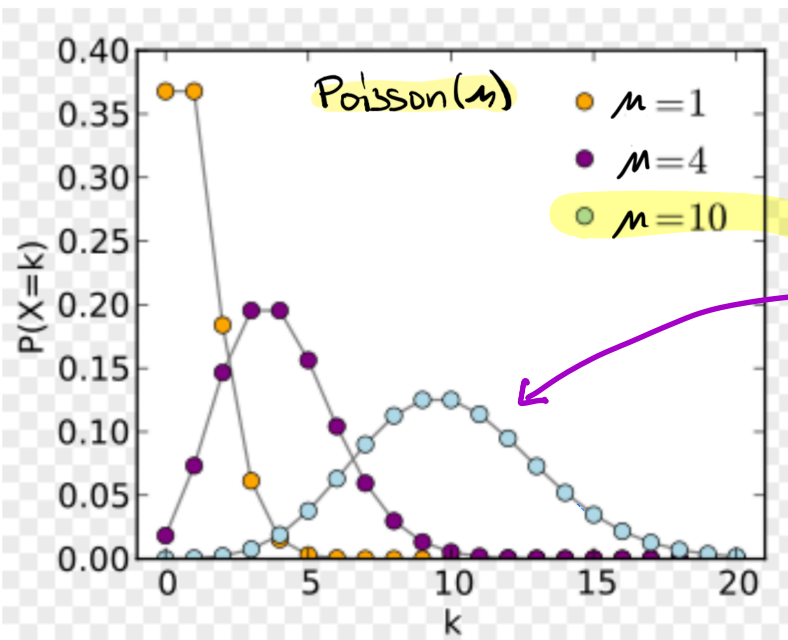
Facts

if $X \sim Pois(1)$, $E(X) = 1$

$Var(X) = 1$

$$E(S_{10}) = E(X_1 + \cdots + X_{10}) = 10E(X_1) = 10$$

$$Var(S_{10}) = Var(X_1 + \cdots + X_{10}) = 10Var(X_1) = 10$$



Pois(10) is a sum of 10 iid Pois(1) and is approx

$N(10, 10)$

μ σ^2

② Sec 3.6 Var of Sum of dependent indicators.

ex

14. A building has 10 floors above the basement. If 12 people get into an elevator at the basement, and each chooses a floor at random to get out, independently of the others, at how many floors do you expect the elevator to make a stop to let out one or more of these 12 people?

X = number of elevator stops.

$p_1 = 1 - \left(\frac{9}{10}\right)^{12}$

a) Find $E(X)$

$X = I_1 + \cdots + I_{10}$

$E(X) = 10 \cdot p_1$

$I_2 = \begin{cases} 1 & \text{if at least 1 person gets off at } 2^{nd} \text{ floor} \\ 0 & \text{else.} \end{cases}$

b) Find $Var(X)$.

$X = I_1 + \cdots + I_{10}$ sum of dependent indicators

$Var(X) = E(X^2) - (E(X))^2$

$E(X^2) = E\left((I_1 + \cdots + I_{10})^2\right) = E\left(\sum_{i,j=1}^{10} I_i I_j\right)$

$I_1 = \begin{cases} 1 & \text{if stop } 1^{st} \text{ floor} \\ 0 & \text{else} \end{cases}$

$I_2 = \begin{cases} 1 & \text{if stop } 2^{nd} \text{ floor} \\ 0 & \text{else} \end{cases}$

$I_1 \cdot I_2 = \begin{cases} 1 & \text{if stop } 1^{st} \text{ and } 2^{nd} \text{ floor} \\ 0 & \text{else} \end{cases}$; j

$\parallel$

I_{12}

$p_{12} = 1 - P\left(\text{no one gets off at } 1^{st} \text{ or } 2^{nd} \text{ floor}\right)$

$= 1 - \left[\left(\frac{9}{10}\right)^{12} + \left(\frac{9}{10}\right)^{12} - \left(\frac{8}{10}\right)^{12}\right]$

$$\begin{array}{c|ccc} & I_1 & \cdots & I_{10} \\ \hline I_1 & I_1 I_{12} & & \\ \vdots & & \ddots & \\ I_{10} & & & I_{10} \end{array}$$

i

$E(X^2) = 10\,E(I_1) + 9 \cdot 10\,E(I_{12}) = 10 p_1 + 9 \cdot 10\, p_{12}$

$Var(X) = E(X^2) - (E(X))^2 = 10 p_1 + 9 \cdot 10\, p_{12} - (10 p_1)^2$

Summary

Variance of sum of dependent i.d. (Identically Distributed) indicators

$X = I_1 + \cdots + I_n$

$p_1 = E(I_1)$

$p_{12} = E(I_{12}) = E(I_1 I_2)$

$E(X) = np_1$

$$Var(X) = \underbrace{np_1 + n(n-1)p_{12}}_{E(X^2)} - \underbrace{(np_1)^2}_{E(X)^2}$$

Variance of sum of i.d. independent indicators

$X = I_1 + \cdots + I_n$

$p_1 = E(I_1)$

$p_{12} = p_1 \cdot p_2 = p_1^2$

$$Var(X) = \underbrace{np_1 + n(n-1)p_1^2}_{E(X^2)} - \underbrace{(np_1)^2}_{E(X)^2} = np_1 - np_1^2$$

$$= np_1(1-p_1)$$

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Stat 134

1. A fair die is rolled 14 times. Let X be the number of faces that appear exactly twice. Which of the following expressions appear in the calculation of $Var(X)$

should be 6.5

a $14 * 13 * \binom{14}{2,2,10}(1/6)^2(1/6)^2(4/6)^{10}$

b $\binom{14}{2}(1/6)^2(5/6)^{12}$

c more than one of the above

d none of the above

X = # faces that appear twice.

$X = I_1 + \cdots + I_6$

$$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ face appears twice} \\ 0 & \text{else} \end{cases}$$

$$p_1 = \binom{14}{2}\left(\tfrac{1}{6}\right)^2\left(\tfrac{5}{6}\right)^{12}$$

$$I_{12} = \begin{cases} 1 & \text{if } 1^{st} \text{ and } 2^{nd} \text{ face appear twice} \\ 0 & \text{else} \end{cases}$$

$$p_{12} = \binom{14}{2,2,10}\left(\tfrac{1}{6}\right)^2\left(\tfrac{1}{6}\right)^2\left(\tfrac{4}{6}\right)^{10}$$

$$Var(x) = \underbrace{np_1 + n(n-1)p_{12}}_{E(x^2)} - \underbrace{(np_1)^2}_{E(x)^2}$$

6. A drawer contains s black socks and s white socks, where s is a positive integer. I pull two socks out at random without replacement and call that my first pair. Then I pull two socks out at random without replacement from the remaining socks in the drawer, and call that my second pair. I proceed in this way till I have s pairs and the drawer is empty.

Let D be the number of pairs in which the two socks are of different colors.

a) Find $E(D)$.

b) Find $Var(D)$.

$$p_1 = \frac{\overset{w}{\binom{s}{1}}\overset{B}{\binom{s}{1}}}{\binom{2s}{2}}$$

Soln

a) $D = I_1 + \dots + I_s$ where $I_k = \begin{cases} 1 & \text{if 2nd pair different} \\ 0 & \end{cases}$

$$\Rightarrow E(D) = s \cdot \frac{\binom{s}{1}\binom{s}{1}}{\binom{2s}{2}}$$

b) $I_{12} = \begin{cases} 1 & \text{if 1st and 2nd pair different} \\ 0 & \text{else} \end{cases}$

$$p_{12} = \frac{\binom{s}{1}\binom{s}{1}}{\binom{2s}{2}} \cdot \frac{\binom{s-1}{1}\binom{s-1}{1}}{\binom{2s-2}{2}}$$

then

$$Var(D) = sp_1 + s(s-1)p_{12} - (sp_1)^2$$