Stat 134 lec 21

Warmup:

Recall $P(T>k) = e^{-\lambda k}$

Let $T \sim Exp(\lambda)$

a) Find $P(T>5) = e^{-5\lambda}$

b) Find $P(T>13 \mid T>8) = \dfrac{P(T>13, T>8)}{P(T>8)} = \dfrac{P(T>13)}{P(T>8)}$

$$= \frac{e^{-13\lambda}}{e^{-8\lambda}} = e^{-5\lambda}$$

Another form of the Memoryless property of Exponential:

$$P(T>k+j \mid T>j) = P(T>k)$$

last time

Sec 4.5 Cummulative Distribution Function (CDF)

The CDF of a RV X is $P(X \le x)$. The survival function is $P(X > x)$.

The CDF or survival function uniquely determine a distribution.

sec 4.2 exponential distribution.

survival function

$T \sim \text{Exp}(\lambda)$ $\quad f(t) = \begin{cases} \lambda e^{-\lambda t} & t \ge 0 \\ 0 & \text{else} \end{cases}$ $\quad$ or $P(T > t) = e^{-\lambda t}$

T = time until 1^{st} arrival in a Poisson Process with rate λ of arrivals

dt here is a small interval right after t

dt here is a small number,

Memoryless Property of the exponential distribution

$$P(T \in dt \mid T > t) = P(0 < T < 0 + dt)$$

ex Cars arrive at a toll booth according to a Poisson process at a rate of λ arrivals per minute. What is the probability there is 1 arrival in dt (a small time interval around t) given that there are no arrivals before time t?

$T \sim \text{Exp}(\lambda)$

$P(T \in dt \mid T > t) = P(1 \text{ arrival in } dt \mid \text{no arrivals before } t)$

dt

t

independence of arrivals

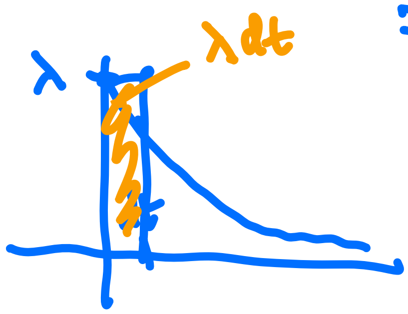
$= P(1 \text{ arrival in } dt)$

$= \dfrac{e^{-\lambda dt}(\lambda dt)}{1!} \approx \lambda dt$

$e^{-\lambda dt} \to 1$ very fast as $dt \to 0$

or use memoryless
property of exponential

$$P(T \in dt \mid T > t) = P(0 < T < 0 + dt)$$
$$= \lambda\, dt$$



Stat 134

Monday October 10 2022

1. GSI Brian and Yiming are each helping a student. Brian and Yiming see students at a rate of λ_B and λ_Y students per hour respectively.
   Let
   $$B = \text{wait time for Brian } \sim Exp(\lambda_B)$$
   $$Y = \text{wait time for Yiming } \sim Exp(\lambda_Y)$$
   What distribution is $T = \min(B, Y)$?
   Hint: compute $P(T > t)$

   a $Exp(\max(\lambda_B, \lambda_Y))$

   b $Exp(\lambda_B - \lambda_Y)$

   c $Exp(\lambda_B + \lambda_Y)$

   d none of the above

a:	The greater the lambda, the smaller the density for T > t. Therefore, we want the greater lambda of the two.

c:	They are independent and since we are working with the minimum. Both of them are greater than t. So we get the probability of Brian's rate greater than t AND Yiming's rate greater than t. Which uding the survival function. We get e^(-lambda(b)-lambda(y)) then you get exp(lambda(b)+lambda(y)).

Today sec 4.2

① Expectation and Variance of $Exp(\lambda)$.

② Gamma distribution.

1

Sec 4.2 Expectation and Variance of Exp(λ)

$T \sim \text{Exp}(\lambda)$

see end of lec notes.

$$E(T) = \int_0^\infty t f(t)\,dt = \lambda \int_0^\infty t e^{-\lambda t}\,dt = \boxed{\frac{1}{\lambda}}$$

see end of lec notes.

$$E(T^2) = \int_0^\infty t^2 f(t)\,dt = \lambda \int_0^\infty t^2 e^{-\lambda t}\,dt = \frac{2}{\lambda^2}$$

$$\text{var}(T) = E(T^2) - E(T)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$

Interpretation of λ:

$\lambda = \frac{1}{E(T)}$ (one over average arrival time)

This makes sense since the instantaneous rate of success is constant for a Poisson process.

So if it takes on avg 5 min to have a success then $\lambda = 1/5$.

ex Let T = waiting time (min) until a blue Honda enters a toll gate starting at noon. The waiting time is on average 2 minutes. Find $P(T > 3)$

$E(T) = 2 \Rightarrow \lambda = 1/2$

$T \sim \text{Exp}(1/2)$

$P(T > 3) = e^{-\frac{1}{2} \cdot 3} = \boxed{e^{-3/2}}$

(2) Gamma Distribution

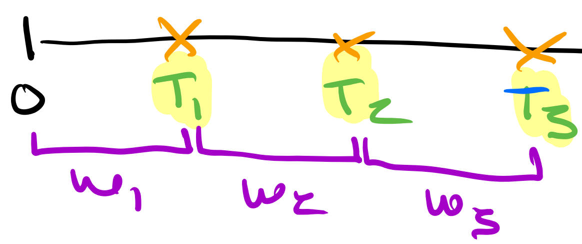
A gamma distribution, $T_r \sim \text{Gamma}(r, \lambda)$, $\lambda > 0$, is a sum of r iid $\text{Exp}(\lambda)$.

$$T_r = W_1 + W_2 + \dots + W_r \quad , \quad W_i \sim \text{Exp}(\lambda)$$

ex T_r = time of the r^{th} arrival of a Poisson Process ($\text{Pois}(\lambda t)$)

ex T_r = time when the r^{th} lightbulb dies.

Picture



$T_1 \sim \text{Gamma}(1, \lambda)$
$T_2 \sim \text{Gamma}(2, \lambda)$
$T_3 \sim \text{Gamma}(3, \lambda)$

dependent but W_1, W_2, W_3 are independent, $\text{Exp}(\lambda)$

$T_1 = W_1$
$T_2 = W_1 + W_2$
$T_3 = W_1 + W_2 + W_3$

Review Poisson Process (a.k.a PRS)

$N_t \sim Pois(\lambda t)$

N_t = # arrivals in time t where the rate of arrivals is λ.

$$P(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

ex Let T_1 = waiting time (min) until a blue Honda enters a toll gate starting at noon. The waiting time is on average 2 minutes. Find the probability 3 blue Hondas arrive in 8 minutes. $\mu = 4$

$t = 8$ $\quad N_8 \sim Pois\left(\frac{1}{2} \cdot 8\right)$

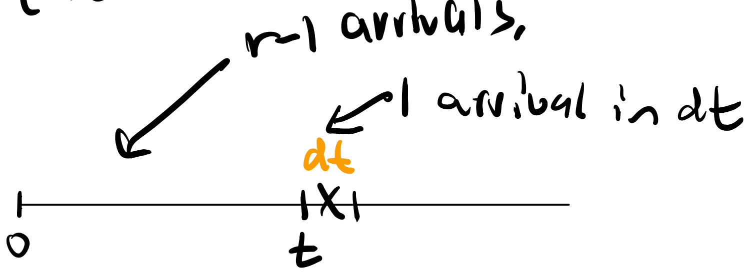
$\lambda = 1/2$

$$P(N_8 = 3) = \boxed{\frac{e^{-4}\, 4^3}{3!}}$$

Back to gamma distribution: $T_r \sim \text{Gamma}(r, \lambda)$

Let $N_t \sim \text{Pois}(\lambda t)$

$P(T_r \in dt)$ means



so $\overbrace{P(T_r \in dt)}^{\approx f(t)\,dt} = P(N_t = r-1) P(1 \text{ arrival in } dt \mid r-1 \text{ arrivals before time } t)$

independence of arrivals in Poisson Process,

$$= P(N_t = r-1) P(1 \text{ arrival in } dt)$$

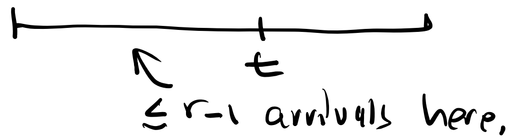
$$\approx \frac{e^{-\lambda t}(\lambda t)^{r-1}}{(r-1)!} \cdot \lambda dt$$

$$= \underbrace{\frac{1}{(r-1)!} \lambda^r t^{r-1} e^{-\lambda t}}_{f(t)} dt \Rightarrow \boxed{f(t) = \frac{1}{(r-1)!} \lambda^r t^{r-1} e^{-\lambda t}}$$

Now, $T_r \sim \text{Gamma}(r, \lambda)$ for $r \in \mathbb{Z}^+$ has density

$$f(t) = \begin{cases} \frac{1}{(r-1)!} \lambda^r t^{r-1} e^{-\lambda t}, & t \geq 0 \\ 0, & \text{else} \end{cases}$$

$N_t \sim \text{Pois}(\lambda t)$



$$P(T_r > t) = P(N_t < r)$$

$$= \sum_{i=0}^{r-1} P(N_t = i)$$

$$= \boxed{\sum_{i=0}^{r-1} \frac{e^{-\lambda t} (\lambda t)^i}{i!}}$$

<u>ex</u> let $T_4 \sim \text{Gamma}(r=4, \lambda=2)$

Find $P(T_4 > 7)$

$$P(T_4 > 7) = P(N_7 < 4)$$

$$N_7 \sim \text{Pois}(\underbrace{2 \cdot 7}_{=14})$$

$$= \boxed{e^{-14}\left(1 + \frac{14}{1!} + \frac{14^2}{2!} + \frac{14^3}{3!}\right)}$$

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λ

1. Suppose customers arrive at a ticket booth at a rate of five per minute, according to a Poisson arrival process. Find the probability that starting from time 0, the $9th$ customer doesn't arrive within 5 minutes:

$= t$

a $\sum_{k=0}^{9} \frac{e^{-25}25^k}{k!}$

b $\sum_{k=0}^{8} \frac{e^{-25}25^k}{k!}$

c $\sum_{k=0}^{8} \frac{e^{-5}5^k}{k!}$

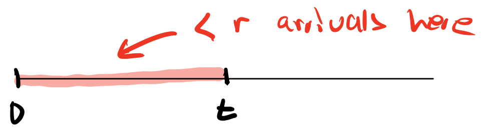
d none of the above

$N_5 \sim \mathrm{Pois}(\lambda t)$, $\lambda t = 5 \cdot 5 = 25$

$P(T_9 > 5) = P(N_5 < 9)$

$$\boxed{\sum_{k=0}^{8} \frac{e^{-25} 25^k}{k!}}$$

$P(T_r > t) = P(N_t < r)$ where $N_t \sim \mathrm{Pois}(\lambda t)$



It is possible to define the Gamma distribution for $r>0$ any real number not just positive integers.

To do this we introduce the Gamma function.

Gamma function $\Gamma(r)$, $r>0$

If $r \in \mathbb{Z}^+$ define $\Gamma(r)=(r-1)!$ where $\Gamma(r)$ is called the gamma function

density of Gamma distribution

For $\lambda=1$, $$f(t) = \underbrace{\frac{1}{\Gamma(r)}}_{\text{const}} \underbrace{t^{r-1} e^{-t}}_{\text{variable}}$$

and $\displaystyle\int_0^\infty f(t)\,dt = 1 \Rightarrow$

$$\frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-t}\,dt = 1$$

$$\Rightarrow \Gamma(r) = \int_0^\infty t^{r-1} e^{-t}\,dt$$

The formula $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t}\, dt$.

is the definition of the gamma function.

You can show that $\Gamma(r) = (r-1)!$

for $r \in \mathbb{Z}^+$

Now, for any $r > 0$ we define $T_r \sim \text{Gamma}(r, \lambda)$

$$f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}, & t \geq 0 \\ 0, & \text{else} \end{cases}$$

This is useful in statistics. We will see later that for $Z \sim N(0,1)$, $Z^2 \sim \text{Gamma}(r = 1/2, \lambda = 1/2)$.

non integer r value.

Appendix

Expectation and Variance of exponential

let $T \sim Exp(\lambda)$

$f(t) = \lambda e^{-\lambda t}$ density

$$E(T) = \lambda \int_0^\infty t e^{-\lambda t} dt$$

I recommend using the tabular method for integration by parts. This works well when the function you are integrating is the product of two expressions, where the n^{th} derivative of one expression is zero.

ex $t^4 e^{3t}$ ✓ good

ex $\sin t \, e^{3t}$ ✗ bad.

To find $\lambda \int_0^\infty t e^{-\lambda t}$:

$\frac{d}{dt}$		$\int$
t	$+$	$e^{-\lambda t}$
1	$-$	$-\frac{e^{-\lambda t}}{\lambda}$
0		$\frac{e^{-\lambda t}}{\lambda^2}$

$$E(T) = \lambda \int_0^\infty t e^{-\lambda t} dt = \lambda \left(t\left(-\frac{e^{-\lambda t}}{\lambda}\right) - 1 \cdot \frac{e^{-\lambda t}}{\lambda^2} \right)\Bigg|_0^\infty$$

$$= 0 - \left(0 - \frac{1}{\lambda}\right)$$

$$= \boxed{\frac{1}{\lambda}}$$

Next find $Var(T) = E(T^2) - E(T)^2$:

$$E(T^2) = \lambda \int_0^\infty t^2 e^{-\lambda t} dt$$

$\frac{d}{dt}$		$\int$
t^2	$+$	$e^{-\lambda t}$
$2t$	$-$	$-\frac{1}{\lambda} e^{-\lambda t}$
2	$+$	$\frac{1}{\lambda^2} e^{-\lambda t}$
0		$-\frac{1}{\lambda^3} e^{-\lambda t}$

$$E(T^2) = \lambda \left(t^2\left(-\frac{1}{\lambda} e^{-\lambda t}\right) - 2t\left(\frac{1}{\lambda^2} e^{-\lambda t}\right) + 2\left(-\frac{1}{\lambda^3} e^{-\lambda t}\right)\right)\Bigg|_0^\infty$$

$$= \frac{2}{\lambda^2}$$

$$\Rightarrow Var(T) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \boxed{\frac{1}{\lambda^2}}$$