Stat 134 lec 22

warmup

Suppose customers are arriving at a ticket booth at rate of five per minute, according to a Poisson arrival process. Find the probability that:

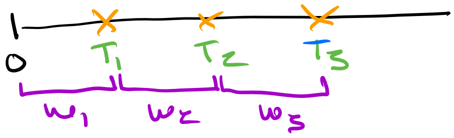
At least one customer arrives within 40 seconds after the arrival of the 13th customer

$$P(\underbrace{T_{14}-T_{13}}_{W \sim Exp(\lambda=5)} < 2/3) = 1 - P(\underbrace{T_{14}-T_{13}}_{W} \geq 2/3) = \boxed{1 - e^{-5\cdot\frac{2}{3}}}$$

Announcement: Monday (lec 23) is a special lecture on moment generating functions (not in textbook).

C

Last time **sec 4.2** **Gamma Distribution**



$T_1 \sim Exp(\lambda), \lambda > 0$ $\qquad f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$ $\leftarrow$ Variable part

$T_r \sim Gamma(r, \lambda), \begin{matrix} \lambda > 0 \\ r > 0 \end{matrix}$ $\qquad f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$ where $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$

$r \in \{1, 2, 3, \dots\}$
then $\Gamma(r) = (r-1)!$

$T_r = W_1 + W_2 + \dots + W_r, \quad W_i \overset{iid}{\sim} Exp(\lambda)$

$E(W_1) = \frac{1}{\lambda} \Rightarrow E(T_r) = \frac{r}{\lambda}$

$Var(W_1) = \frac{1}{\lambda^2} \Rightarrow Var(T_r) = \frac{r}{\lambda^2}$

Ex A random variable X has non negative values and density $cx^4 e^{-3x}$ for $0 \leq x < \infty$, and some constant c.
What distribution is X? $X \sim Gamma(5, 3)$ $\quad c = \frac{3^5}{4!}$
Find $Var(X) = \frac{5}{9}$

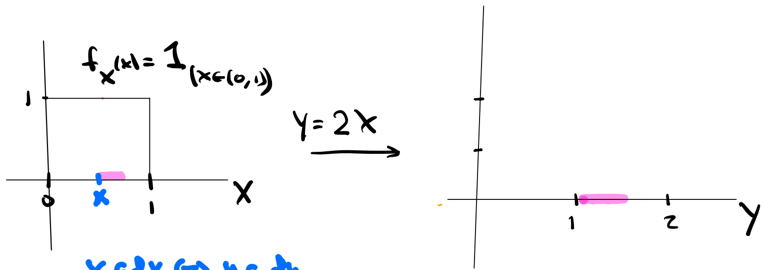
Today Sec 4.4 (skip 4.3)

① Change of Variable formula for densities.

① Sec 4.4 Change of Variable Formula for densities

ex

Let $X \sim U(0,1)$. What distribution is $Y = 2X$?



$x \in dx \Leftrightarrow y \in dy$

$$P(Y \in dy) = P(X \in dx)$$

$$f_Y(y)\,dy = f_X(x)\,dx$$

$$\Rightarrow f_Y(y) = f_X(x) \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

$$= \frac{1}{\frac{dy}{dx}} f_X(x) = \frac{1}{2} \cdot \mathbb{1}_{(x \in (0,1))}$$

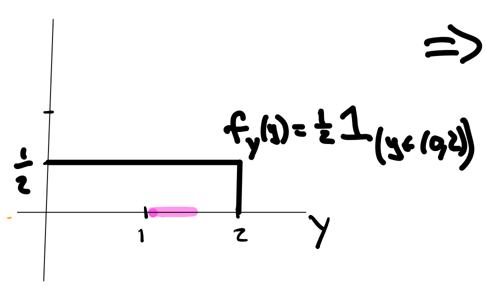
$$= \frac{1}{2} \mathbb{1}_{\left(\frac{y}{2} \in (0,1)\right)}$$

$$= \frac{1}{2} \mathbb{1}_{(y \in (0,2))}$$

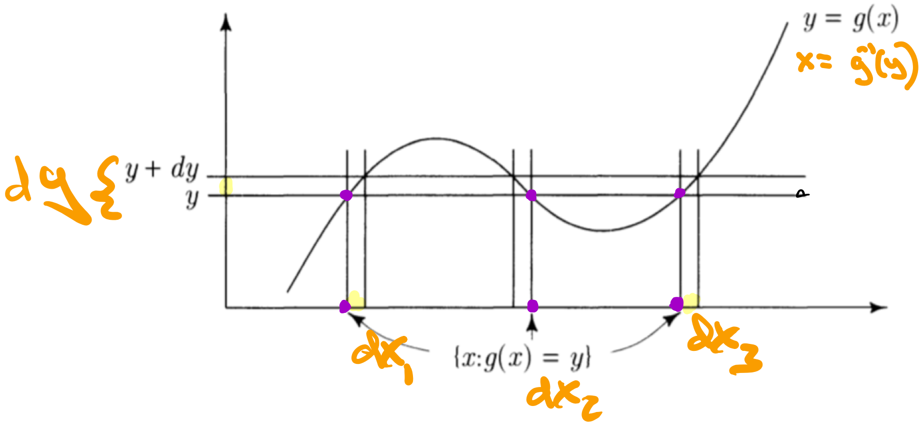
$$= \frac{1}{2} \text{ for } y \in (0,2)$$

zero else

$\Rightarrow Y \sim \text{Unif}(0,2)$



More generally if X has density $f_X(x)$ lets find the density of $Y = g(X)$



mutually exclusive events.

$Y \in dy$ iff $X \in dx_1$ or $X \in dx_2$ or $X \in dx_3$

$$P(Y \in dy) = P(X \in dx_1) + P(X \in dx_2) + P(X \in dx_3)$$

$$f_Y(y)\,dy = f_X(x_1)\,dx_1 + f_X(x_2)\,dx_2 + f_X(x_3)\,dx_3$$

$$f_Y(y) = f_X(x_1)\frac{dx_1}{dy} + f_X(x_2)\frac{dx_2}{dy} + f_X(x_3)\frac{dx_3}{dy}$$

$$= \frac{f_X(x_1)}{g'(x_1)} + \frac{f_X(x_2)}{|g'(x_2)|} + \frac{f_X(x_3)}{g'(x_3)}$$

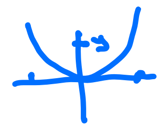
evaluated at $x = g^{-1}(y)$

$P(X \in dx_2) \geq 0$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{g'(x)}$$

Thm (P307) **Change of Variable Formula for densities**

Let X be a continuous RV with density $f_X(x)$.



Let $Y = g(X)$ have a derivative that is zero at only finitely many pts,

$y = x^2$

$x = \pm\sqrt{y}$

then $$f_Y(y) = \sum_{\{x \mid g(x) = y\}} \frac{f_X(x)}{|g'(x)|}\Bigg|_{x = g^{-1}(y)}$$

replace x with $g^{-1}(y)$

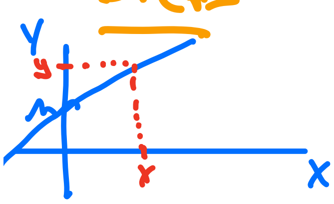
ex

let $X = N(0,1)$, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

* — we will show later in the semester that this is a density

Find the density of $Y = \sigma X + \mu$ where $\sigma > 0$, $\mu \in \mathbb{R}$

Steps



1) Find $g(x) = \sigma x + \mu$
2) Find $g'(x) = \sigma$
3) Find $x = g^{-1}(y) = \frac{y-\mu}{\sigma}$
4) Find $f_Y(y) = \frac{f_X(x)}{|g'(x)|}\Bigg|_{x = \frac{y-\mu}{\sigma}}$

$$= \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}{\sigma}\Bigg|_{x = \frac{y-\mu}{\sigma}}$$

the method the normal density from lecture 5 page 9

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$$

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Stat 134

1. Let $X \sim Unif(0,1)$. The density of $Y = X^2$ is:

   a $f(y) = \frac{1}{\sqrt{y}}$ for $y \in (0,1)$, zero else.

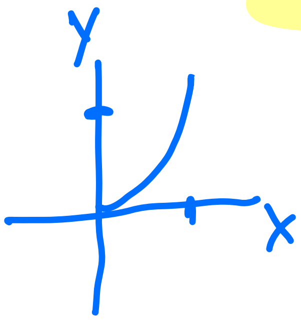
   b $f(y) = \frac{1}{2\sqrt{y}}$ for $y \in (0,1)$, zero else.

   c $f(y) = 1$ for $y \in (0,1)$, zero else.

   d none of the above

$$f_Y(y) = \sum_{\{x \mid g(x)=y\}} \frac{f_X(x)}{|g'(x)|}\Bigg|_{x=g^{-1}(y)}$$

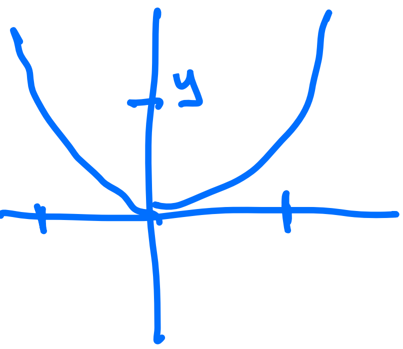
evaluated at $x=g^{-1}(y)$



$$f_Y(y) = \frac{1}{|2x|}\mathbb{1}_{x\in(0,1)}\Bigg|_{x=\sqrt{y}}$$

$$= \frac{1}{2\sqrt{y}}\mathbb{1}_{\sqrt{y}\in(0,1)} = \boxed{\frac{1}{2\sqrt{y}}\mathbb{1}_{y\in(0,1)}}$$

How does the answer change if $X \sim Unif(-1,1)$?



$g(x) = x^2$

$g'(x) = 2x$

$f_X(x) = \frac{1}{2}\mathbb{1}_{x\in(-1,1)}$

$$f_Y(y) = \frac{\frac{1}{2} 1_{x \in (-1,1)}}{|2x|} \Bigg|_{x = \pm\sqrt{y}}$$

$$= \frac{1}{4\sqrt{y}} \left(1_{\sqrt{y} \in (-1,1)} + 1_{-\sqrt{y} \in (-1,1)} \right)$$

$$\sqrt{y} \in (-1,1) \;=\; \sqrt{y} \in (0,1) \;=\; y \in (0,1)$$

$$-\sqrt{y} \in (-1,1) \;=\; -\sqrt{y} \in (-1,0) \;=\; y \in (0,1)$$

$$= \frac{1}{2\sqrt{y}} \cdot 1_{y \in (0,1)}$$

ex (extra problem)

(3 pts) Suppose the random variable X, which measures the magnitude of an earthquake (on the Richter scale) in the Bay Area, follows the Exponential (λ) distribution. Since the Richter scale is logarithmic, we want to study the distribution of the total energy of earthquakes. Find the distribution of $Y = e^X$.

$\leftarrow g(x)$

Change of variable formula:

$$f_Y(y) = \sum_{\{x \mid g(x)=y\}} \frac{f_X(x)}{|g'(x)|} \quad \text{evaluated at } x = g^{-1}(y)$$

Find $g(x) = e^x$

$g'(x) = e^x$

$g^{-1}(x) = \ln y$

$$f_Y(y) = \frac{\lambda e^{-\lambda x}}{e^x}\Bigg|_{x=\ln y} = \lambda e^{(-\lambda-1)x}\Bigg|_{x=\ln y}$$

$$= \lambda e^{(-\lambda-1)\ln y} = \lambda \big(\underbrace{e^{\ln y}}_{y}\big)^{(-\lambda-1)} = \boxed{\lambda y^{(-\lambda-1)}, \quad y>1}$$