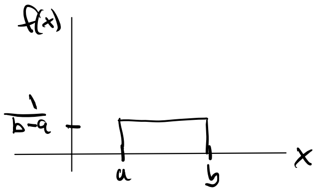
Stat 134 lec 25

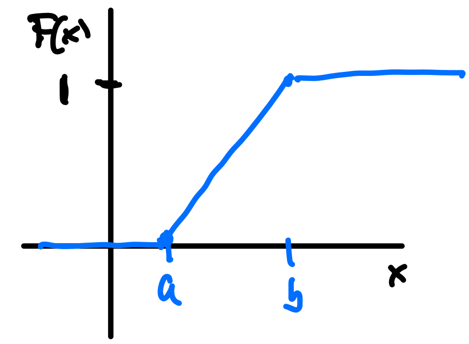
Warmup

Recall that the commulative distribution function (CDF) for a RV X is $F(x) = P(X \le x)$.

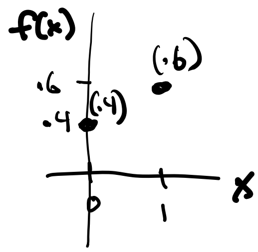
Draw the CDF for each of the distributions below:

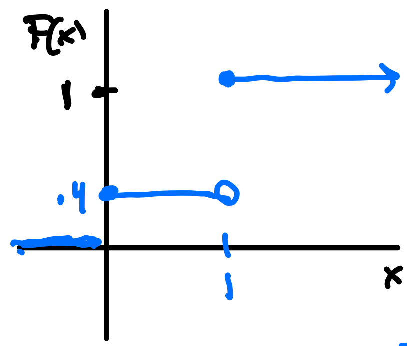
ex $X \sim Unif(a,b)$



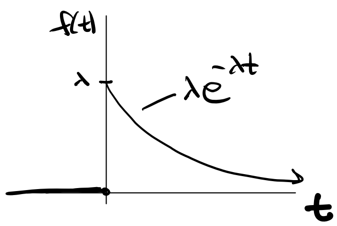


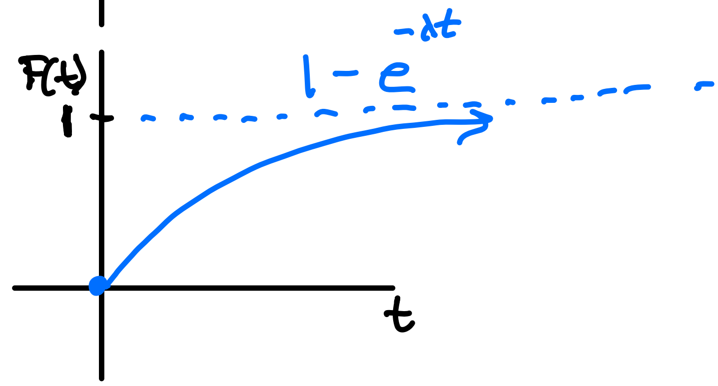
ex $X \sim Bernoulli(p=.6)$





ex $T \sim Exp(\lambda)$





last time

MGF

(1) Key Properties of MGF

(a) If an MGF exists in an interval containing zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

last time

(b) If X and Y are independent RVs,

$M_{X+Y}(t) = M_X(t) M_Y(t)$.

Proved in MGF HW.

(c) If $M_X(t) = M_Y(t)$ for all t in an interval around 0 then $F_X(z) = F_Y(z)$ (i.e. X and Y have the same distribution).

Variable part of density

A density can be written as

$$f(t) = c\, h(t)$$

variable part.

constant

$$1 = \int_{-\infty}^{\infty} f(t)\,dt = c \int_{-\infty}^{\infty} h(t)\,dt \Rightarrow c = \frac{1}{\int_{-\infty}^{\infty} h(t)\,dt}$$

So you can figure out the density from its variable part.

Concept test from last time:

Let X be the standard normal RV. The distribution of $Y = X^2$ is:

a Gamma

b Uniform

c Normal

d none of the above

a	Formula for standard normal is e^(-0.5x^2). Using change of variable formula we get f(y)= y^(-0.5)e^(-0.5y), which is the variable part of Gamma(0.5, 0.5)

details:

$$g(x) = x^2$$

$$g'(x) = 2x$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$f(y) = \left.\frac{f(x)}{|2x|}\right|_{x=\sqrt{y}} + \left.\frac{f(x)}{|2x|}\right|_{x=-\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-y/2}}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} \frac{e^{-y/2}}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-\frac{1}{2}y}$$

variable part of Gamma $(1/2, 1/2)$

Today

① sec 4.5 Find CDF of a mixed distribution

② sec 4.5 Using CDF to find $E(X)$

① sec 4.5 CDF of mixed distributions

eg (Mixed distribution)

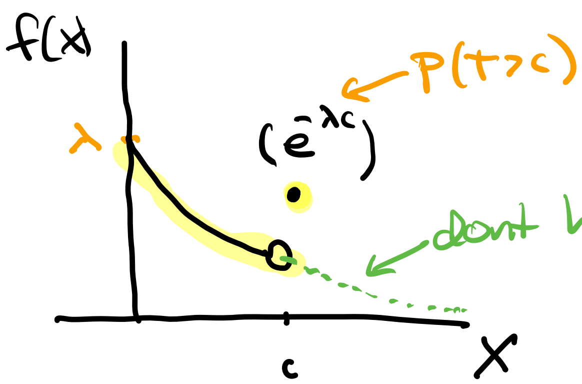
$T \sim Exp(\lambda)$

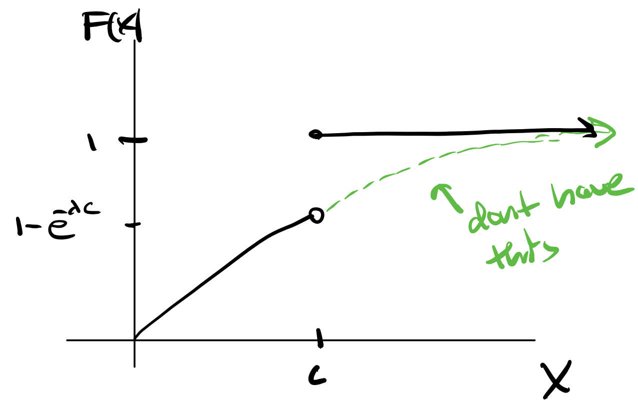
$c > 0$

$$X = \begin{cases} T & \text{if } x < c \\ c & \text{if } x = c \end{cases}$$

$X = \min(T, c)$

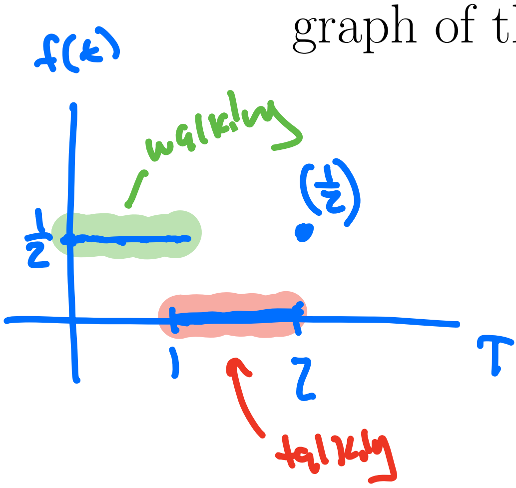
"T killed at c"

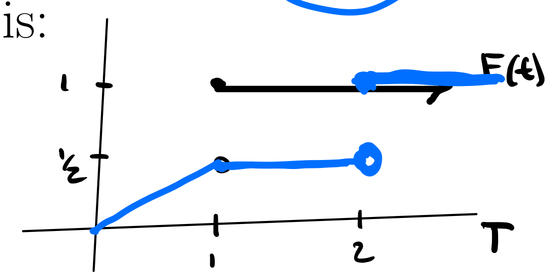




ex

Suppose you are trying to discretly leave a party. Your time to leave is uniform from 0 to 2 minutes. However, if your walk to the exit takes more than 1 minute, you run into a friend at the door and and must spend the full 2 minutes to leave. Let T represent the time it takes you to leave. True or false, the graph of the cdf of T is:



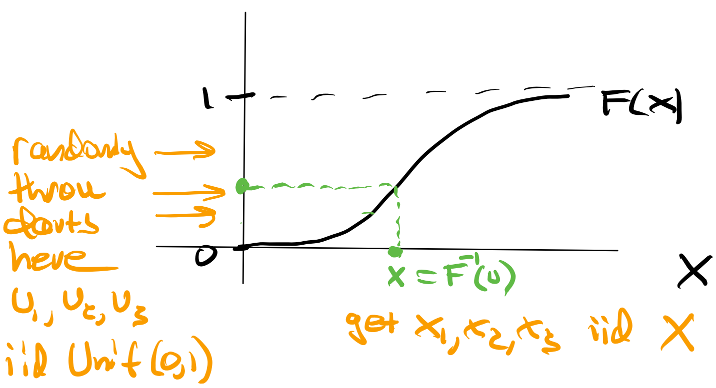


② Sec 4.5 Using CDF to find $E(X)$ for $X \geq 0$

see #9 p324

Inverse distribution function, $F^{-1}(u)$

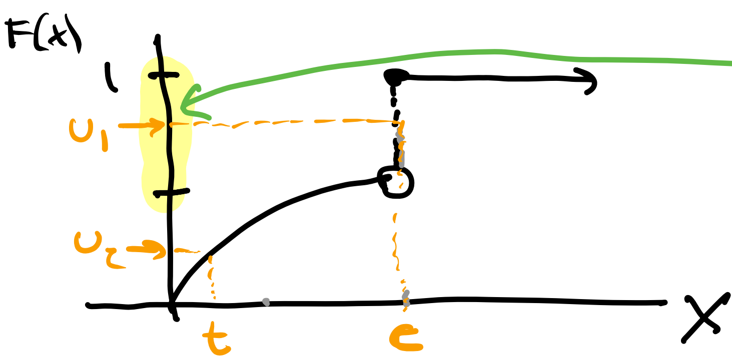
Let X have CDF $F(x)$.



$$\boxed{F^{-1}(u) = x}$$

Note: doesn't have to be continuous RV.

ex $X = \min(T, c)$, $T \sim Exp(\lambda)$

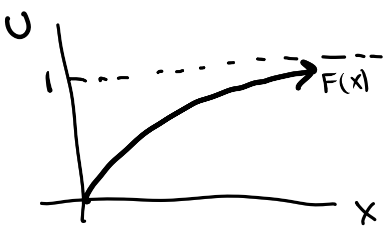


This yellow region gets assigned the single value c.

Thm (p322) — Proof at end of lecture.

Let X have CDF F.

Then the RV $F^{-1}(U) = X$

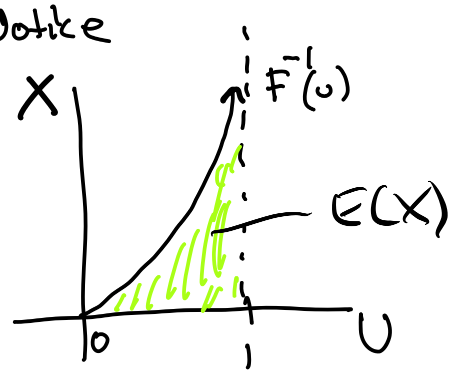


How is this useful to us finding $E(X)$?

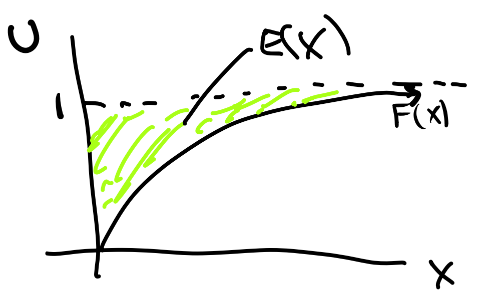
1 since $U \sim \text{Unif}(0,1)$

$$E(X) = E(F^{-1}(U)) = \int_0^1 F^{-1}(u) f_U(u)\, du$$

Notice



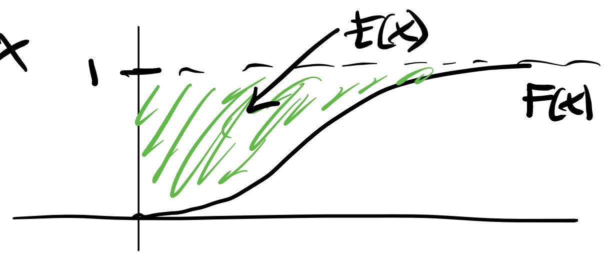
Now reflect the above graph about the diagonal $y = x$



We can find the shaded region by integrating $1-F(x)$ with respect to x:

Thm Let X be a pos. random variable, with CDF F. (continuous, discrete, mixed),

$$E(X) = \int_0^\infty (1-F(x))\,dx$$



ex $T \sim \text{expon}(\lambda)$

$$F_T(t) = 1 - e^{-\lambda t}$$

Calculate $E(T)$.

$$E(T) = \int_0^\infty (1-F(t))\,dt = \int_0^\infty e^{-\lambda t}\,dt = -\frac{1}{\lambda} e^{-\lambda t}\Big|_0^\infty = \boxed{\frac{1}{\lambda}}$$

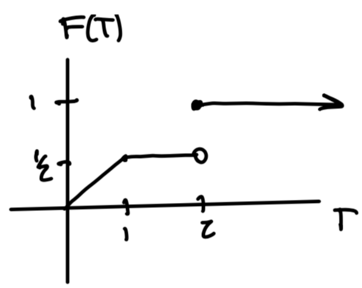
It is sometimes easier to calculate $E(X)$ using the cdf (avoid doing integration by parts): $E(T) = \int_0^\infty t\lambda e^{-\lambda t}\,dt$

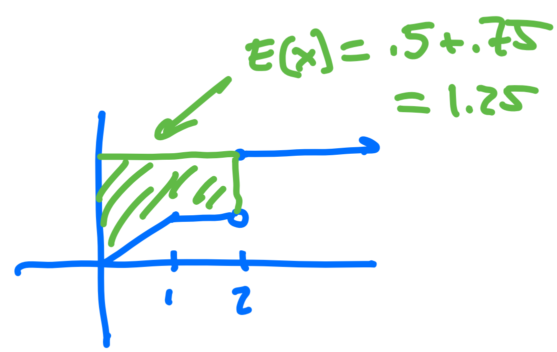
tinyurl.com/mar17-2023

Stat 134

Friday October 21 2022

1. Suppose you are trying to discretely leave a party. Your time to leave is uniform from 0 to 2 minutes. However, if your walk to the exit takes more than 1 minute, you run into a friend at the door and must spend the full 2 minutes to leave. Let T represent the time it takes you to leave.





Your expected time to leave, $E(T)$, is:

a 0.5 min

b 0.75 min

c 1.25 min

d none of the above

Appendix

See p 322 in book

Claim for any CDF F

$X = F^{-1}(U)$ is a RV with cdf F.

Unif(0,1)

Proof/ let $X = F^{-1}(U)$

$$F_X(x) = P(X \le x)$$

we will show $F_X = F$

$$= P(F^{-1}(U) \le x)$$

$$= P(F \cdot F^{-1}(U) \le F(x))$$

since F is increasing

$$= P(U \le F(x))$$

$$= F(x)$$

since $P(U \le u) = u$

$\square$