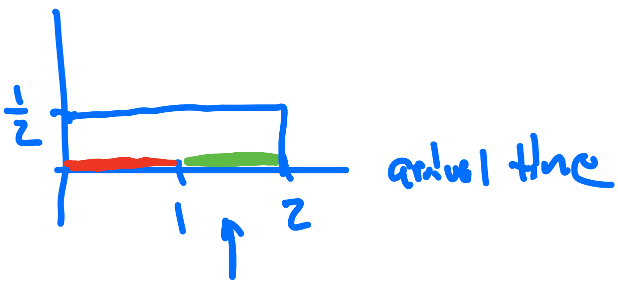
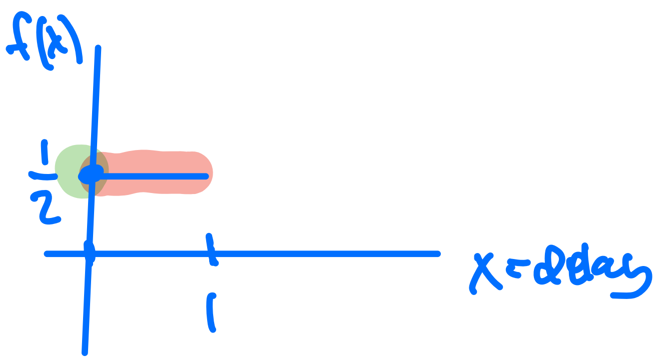
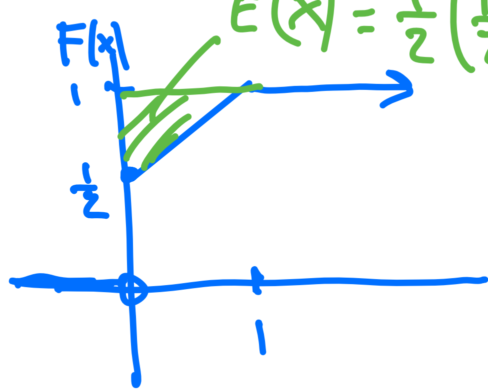
warmup:

Suppose stop lights at an intersection alernately show green for one minute, and red for one minute (no yellow). Suppose a car arrives at the lights at a time distributed uniformly from 0 to 2 minutes. Let X be the delay of the car at the lights (assuming there is only one car on the road). Graph the density and the cdf of X. Also find E(X)







$$E(X) = \frac{1}{2}\left(\frac{1}{2}\cdot 1\right) = \frac{1}{4}$$

Last time

sec 4.5 Expectation of a nonnegative RV using CDF

$$E(X) = \int_0^\infty (1-F(x))\,dx$$

ex let $X \sim \text{Geom}\left(\frac{1}{2}\right)$

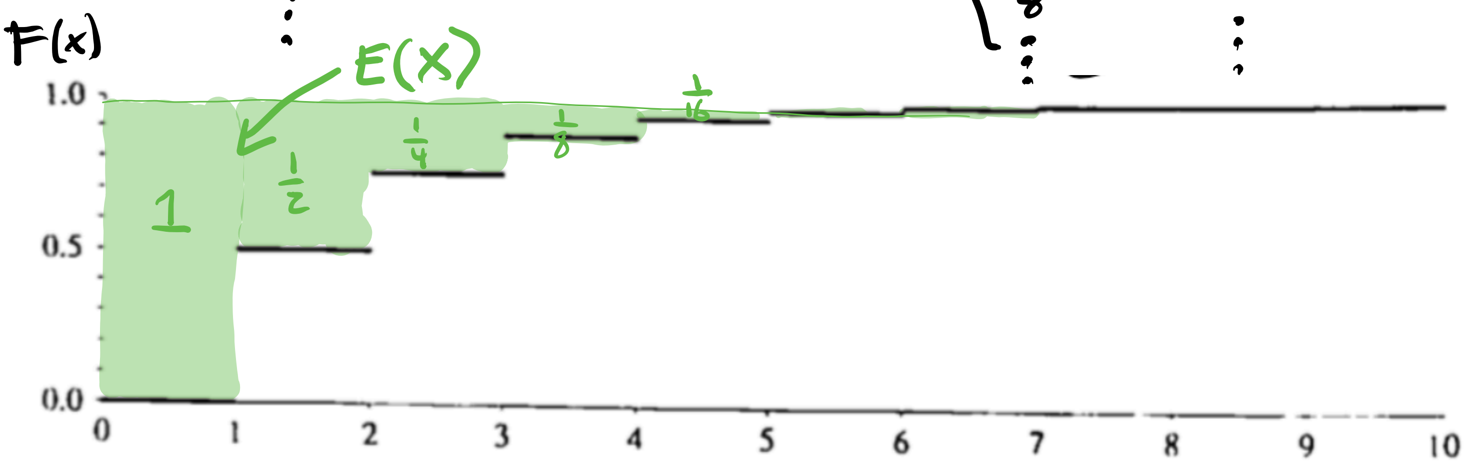
$P(X=1) = 1/2$

$P(X=2) = \frac{1}{2}\cdot\frac{1}{2} = 1/4$

Picture $P(X=3) = \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} = 1/8$

$\vdots$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 1 \\ 1/2 & 1 \le x < 2 \\ \frac{3}{4} & 2 \le x < 3 \\ \frac{7}{8} & 3 \le x < 4 \\ \vdots & \vdots \end{cases}$$



$$E(X) = \int_0^\infty (1-F(x))\,dx$$

$$E(X) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \boxed{2}$$

p

$$= \sum_{j=0}^{\infty} \underbrace{P(X>j)}_{= q^j} = \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j \quad \leftarrow \text{tail sum formula.}$$

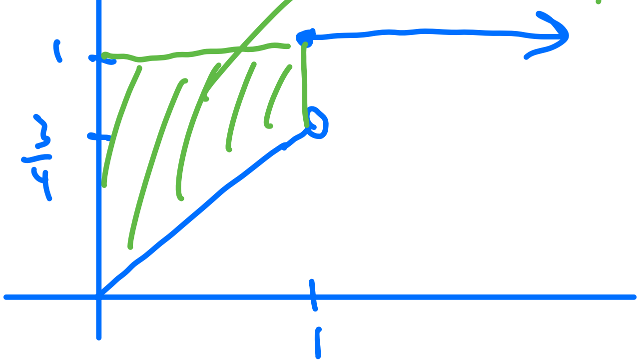
ex

A random variable X has CDF

$$F(x) = \begin{cases} \frac{3x}{4} & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

The expectation of X is:

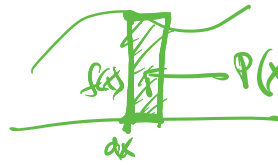
$E(X) = \frac{1}{2}\left(\frac{3}{4}\right) + \frac{1}{4} = \frac{5}{8}$



① Overview of what we have learned since the midterm.

② sec 4.6 Order statistics

① Overview



$P(X \in dx) \approx f(x)dx$

Chap 4

single variable unconditional prob

density of distribution

change of variable formula for densities,

expectation

continuous distributions

- uniform
- exponential/gamma
- order statistics / beta

MGF - useful tool

- calculate moments
- identify a distribution by its MGF

CDF / mixed distributions

calculating expectation from cdf.

Chap 5

multiple variable unconditional prob

joint distributions

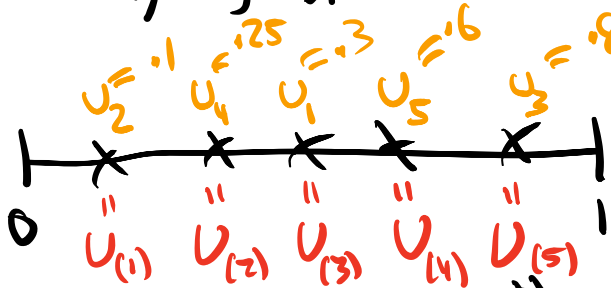
Chap 6

multiple variable conditional prob.

dependence

① **Sec 4.6** order statistic of $U(0,1)$

let $U_1, \ldots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$



throw 5 darts at $[0,1]$.

let $U_{(k)}$ = called the k^{th} order statistic

$= k^{th}$ value of $U_1, \ldots, U_n$ sorted smallest to biggest.

(assuming no ties)

ex

$$U_{(1)} = \min(U_1, \ldots, U_n)$$
$$U_{(n)} = \max(U_1, \ldots, U_n)$$

Review counting

You have 3 red, 2 green and 5 blue marbles, How many orderings of these 10 marbles are there?

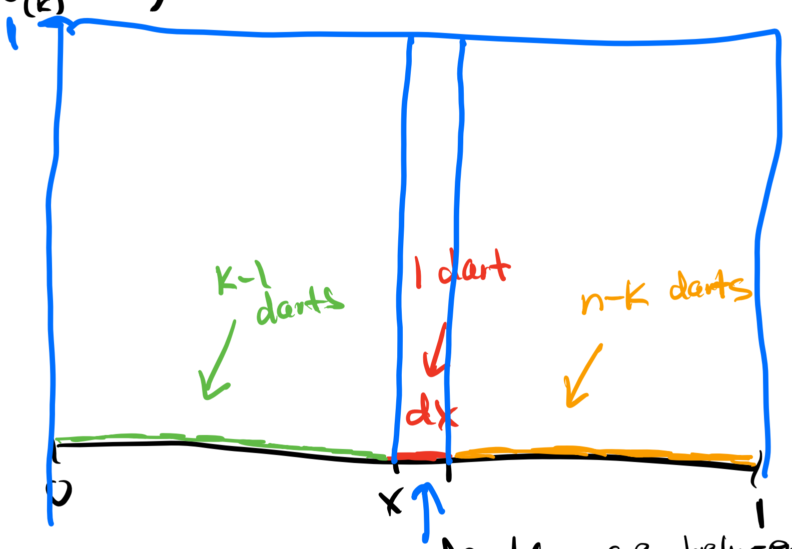
ex rrrggbbbbb
grrrgbbbbb
ggrrrbbbbb
⋮

$$\binom{10}{3,2,5} = \binom{10}{3}\binom{7}{2}\binom{5}{5}$$
$$= \frac{10!}{3!\,2!\,5!}$$

Next, find density of $U_{(k)}$

write $P(U_{(k)} \in dx) \approx f(x)dx$

Picture



$U_{(k)} \in dx$ means that $k-1$ darts are between 0 and x, and one is in dx, and $n-k$ darts are between x and 1

$$P(U_{(k)} \in dx) = P\left(k-1 \text{ darts} \in (0,x),\ 1 \text{ dart} \in dx,\ n-k \text{ darts} \in (x,1)\right)$$

$$= P\left(k-1 \text{ darts} \in (0,x)\right) \cdot P\left(1 \text{ dart} \in dx \mid k-1 \text{ darts} \in (0,x)\right)$$

$$\cdot P\left(n-k \text{ darts} \in (x,1) \mid 1 \text{ dart} \in dx,\ k-1 \text{ darts} \in (0,x)\right)$$

$$= \binom{n}{k-1} x^{k-1} \binom{n-k+1}{1} 1\,dx \binom{n-k}{n-k} (1-x)^{n-k}$$

$$= \underbrace{\binom{n}{k-1,\,1,\,n-k} x^{k-1} (1-x)^{(n-k+1)-1}}_{f_{U_{(k)}}(x)} dx$$

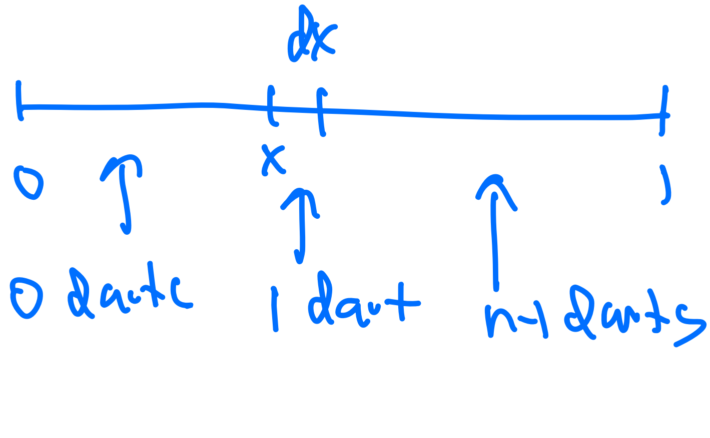
$$\Rightarrow f_{U_{(k)}}(x) = \binom{n}{k-1,\, 1,\, n-k} x^{k-1} (1-x)^{(n-k+1)-1} \quad \text{for } 0<x<1$$

Ex Let $U_1, \dots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$

Find the density of $\underline{U_{(1)}}$

$P(U_{(1)} \in dx)$

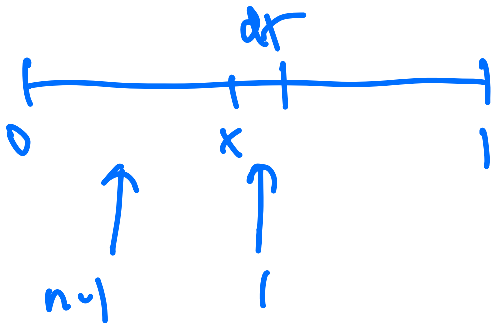
$$f_{U_{(1)}}(x) = \binom{n}{0,\, 1,\, n-1} (1-x)^{n-1}, \quad 0<x<1$$



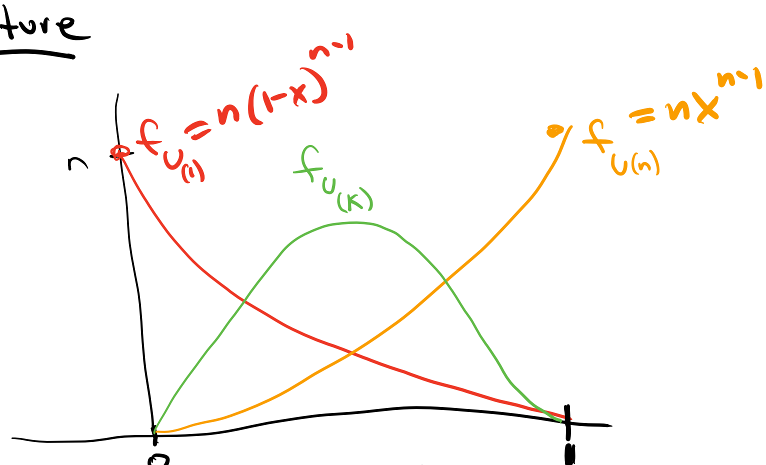
Ex Let $U_1, \ldots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$

Find the density of $U_{(n)}$

$$f_{U_{(n)}}(x) = \binom{n}{n-1, 1, 0} x^{n-1}, \quad 0<x<1$$



Picture



Order statistic of $U(0,1)$ provides a family of densities on the unit interval.

tinyurl.com/mar20-2023

Stat 134

1. $x^2(1-x)^4$ for $0 < x < 1$ is the variable part of the density of what random variable?

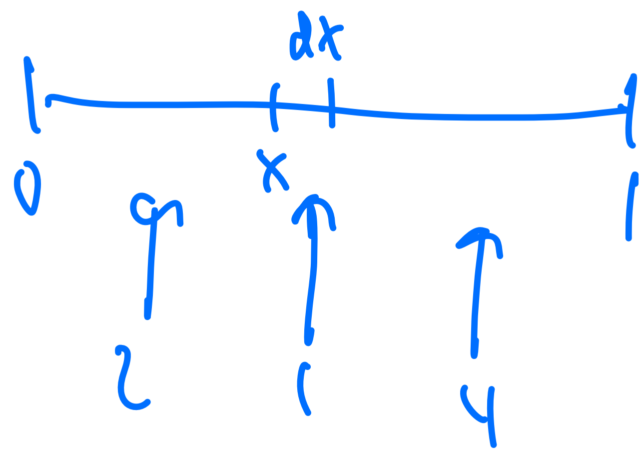
   a $U_{(3)}$ of n=6 darts

   b $U_{(2)}$ of n=7 darts

   c $U_{(1)}$ of n=7 darts

   d none of the above

$U_{(3)}$ out of 7



ex: Let $X_1,\dots,X_n \overset{iid}{\sim} \mathrm{Unif}(0,b)$.

Find the density of $Y=\max(X_1,\dots,X_n)$.

$P(Y\in dy) = f(y)\,dy$

$$P(Y\in dy) = f_Y(y) = \underbrace{\binom{n}{n-1,1,0}}_{=n}\left(\frac{y}{b}\right)^{n-1}\left(\frac{1}{b}dx\right)$$

$$\Rightarrow f_Y(y) = \frac{n y^{n-1}}{b^n}, \quad 0<y<b$$