Stat 134 lec 30

warmup 11:00-11:10

ex (5.2.9a)

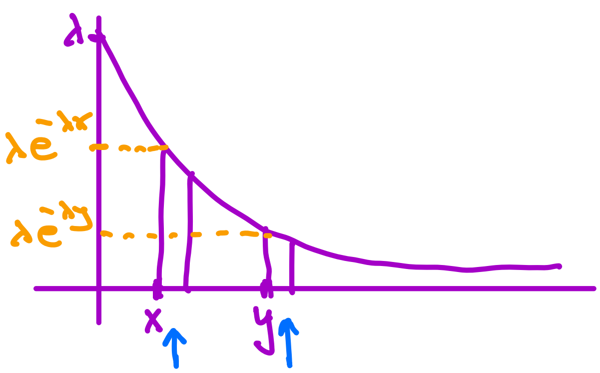
$S, T \overset{iid}{\sim} Exp(\lambda)$ $\left(f_S(s) = \lambda e^{-\lambda s} \right)$

$X = \min(S,T)$ ← 1^{st} ordered statistic of $Exp(\lambda)$

$Y = \max(S,T)$ ← 2^{nd} ordered statistic of $Exp(\lambda)$

Find the joint density of X and Y

Picture



$$P(X \in dx, Y \in dy) = \binom{2}{1} \lambda e^{-\lambda x} dx \cdot \binom{1}{1} \lambda e^{-\lambda y} dy$$

$$= 2\lambda^2 e^{-\lambda(x+y)} dx\,dy$$

$$\Rightarrow \boxed{f(x,y) = 2\lambda^2 e^{-\lambda(x+y)} \quad \text{for} \quad 0 \le x \le y}$$

Earlier material

$T \sim Exp(\lambda)$, $cT \sim Exp\left(\frac{\lambda}{c}\right)$

Competing exponentials

$T_1 \sim Exp(\lambda_1)$, $T_2 \sim Exp(\lambda_2) \Rightarrow P(T_1 < T_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$

Properties of std normal Z		Proved it
$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$		no
$E(Z) = 0$	Even though Z is symmetric around zero it is possible $E(Z)$ is undefined. ex Cauchy distribution	no
$SD(Z) = 1$		no

Let $X, Y \overset{iid}{\sim} N(0,1)$ with density $\phi(x) = ce^{-\frac{1}{2}x^2}$, $c>0$

$f(x,y) = \phi(x)\phi(y) = c^2 e^{-\frac{1}{2}(x^2+y^2)}$ for $c>0$

We still need to show that $c = \frac{1}{\sqrt{2\pi}}$, $E(X) = 0$, $SD(X) = 1$.

Last time

sec 5.2 Marginal density $f_Y(y) = \int_{x=-\infty}^{x=\infty} f(x,y)\,dx$

Today

(1) Sec 5.2 Marginal Densities

(1) Sec 5.2 Expectation $E(g(X,Y))$

(2) sec 5.3 Rayleigh distribution

① Sec 5.2 Marginal Densities

Stat 134

Friday November 8 2019

1. S and T are i.i.d. Exp(λ). $X = \text{Min}(S,T)$ and $Y = \text{Max}(S,T)$. The joint density is $f(x,y) = 2\lambda^2 e^{-\lambda(x+y)}$. The marginal density of Y is:

   a $\lambda(1-e^{-\lambda y})e^{-\lambda y}$ for $y > 0$

   b $2\lambda(1-e^{-\lambda y})e^{-\lambda y}$ for $y > 0$

   c $2\lambda(1-e^{-\lambda y})$ for $y > 0$

   d none of the above

$$f_Y(y) = 2\lambda^2 e^{-\lambda y} \underbrace{\int_{x=0}^{x=y} e^{-\lambda x}\,dx}_{} = \boxed{2\lambda(1-e^{-\lambda y})e^{-\lambda y} \text{ for } y>0}$$

$$\frac{e^{-\lambda x}}{-\lambda}\Bigg|_0^y = \frac{1-e^{-\lambda y}}{\lambda}$$

Or

$$F(y) = P(Y<y) = P(S<y, T<y) = (P(S<y))^2 = (1-e^{-\lambda y})^2$$

$$f(y) = \frac{d}{dy}F(y) = 2(1-e^{-\lambda y})(e^{-\lambda y})\lambda = \boxed{2\lambda(1-e^{-\lambda y})(e^{-\lambda y}) \text{ for } y>0}$$

② Sec 5.2 Expectation $E(g(X,Y))$

Let (X,Y) have joint density $f(x,y)$.
and $g(X,Y)$ be a function of X, Y.

Define

$$E(g(x,y)) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} g(x,y) f(x,y)\, dx\, dy$$

ex

joint density

$$f(x,y) = \begin{cases} 30\,(y-x)^4 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

$X = U_{(1)}$
$Y = U_{(6)}$

Find

$$E(Y) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} y\, f(x,y)\, dx\, dy$$

$g(X,Y) = Y$

we know $Y = U_{(6)} \sim \text{Beta}(6,1) \Rightarrow E(Y) = \frac{6}{6+1} = \boxed{\frac{6}{7}}$

k $n-k+1 = 6-6+1 = 1$

See appendix to notes.

③ **Sec 5.3** **Rayleigh Distribution**

let $T \sim \text{Exp}(\frac{1}{2})$, $f(t) = \frac{1}{2}e^{-\frac{1}{2}t}$, $t > 0$

$R = \sqrt{T}$ ⟵ R is called the Rayleigh Distribution

write $R \sim \text{Ray}$

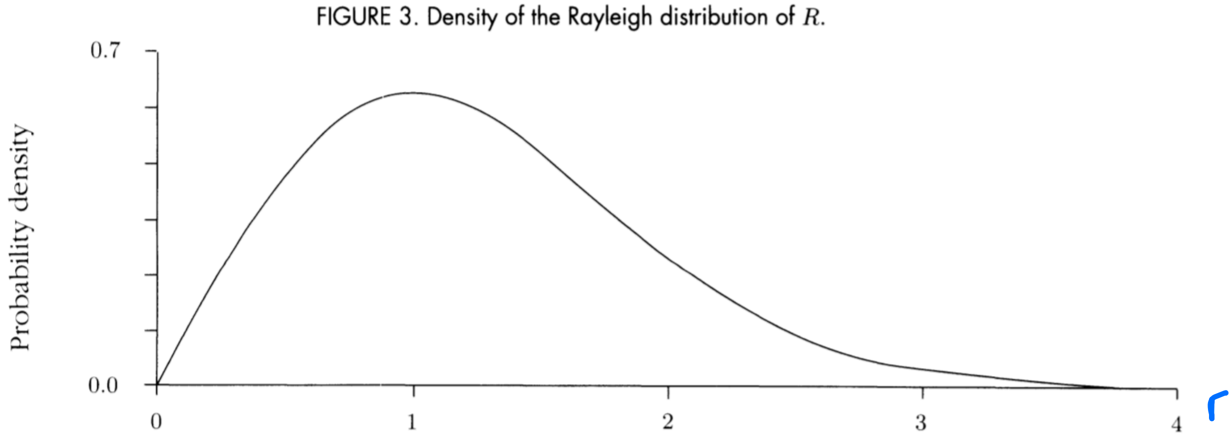
Find $f_R(r)$.

$$f_R(r) = \frac{f_T(t)}{\left|(\sqrt{t})'\right|}\Bigg|_{t=r^2}$$

$$= \frac{\frac{1}{2}e^{-\frac{1}{2}r^2}}{\frac{1}{2r}} = \boxed{re^{-\frac{1}{2}r^2}, \quad r > 0}$$

$\sqrt{t}$

FIGURE 3. Density of the Rayleigh distribution of R.



Note:

$$P(R > r) = P(R^2 > r^2) = P(\overset{\sim \text{Exp}(\frac{1}{2})}{T} > r^2)$$

$$= e^{-\frac{1}{2}r^2}$$

So $\boxed{F_R(r) = 1 - e^{-\frac{1}{2}r^2}, \quad r > 0}$

$$f_R(r) = \frac{d}{dr} F_R(r)$$

$$= 0 + \frac{1}{2} e^{-\frac{1}{2}r^2} \cdot 2r = \boxed{\begin{array}{c} r e^{-\frac{1}{2}r^2} \\ r > 0 \end{array}}$$

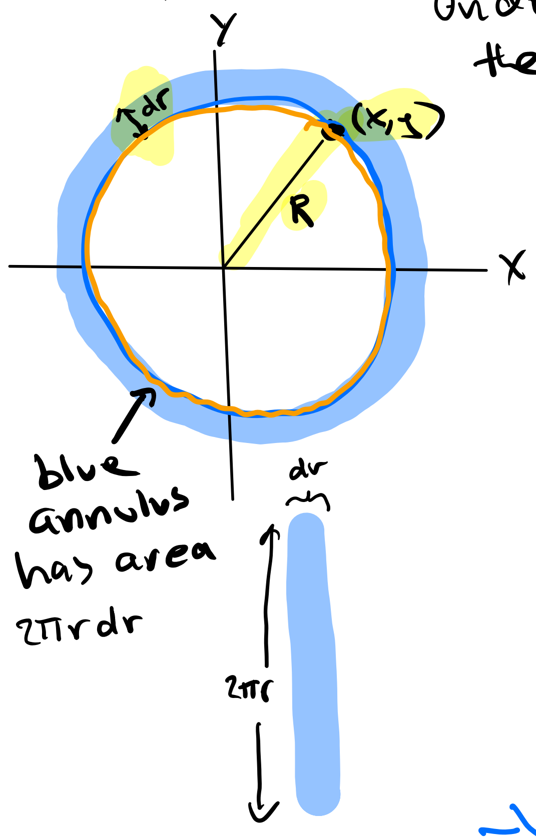
The Rayleigh distribution will help us find the density of the standard normal:

For $X, Y \overset{iid}{\sim} N(0,1)$

Let $R = \sqrt{X^2 + Y^2}$

We will show that $R \sim \text{Ray}$

$P(R \in dr)$ is the volume of the cylinder under $f(x,y) = c^2 e^{-\frac{1}{2}(x^2+y^2)}$ over the shaded blue annulus.



$P(R \in dr) \approx$ height of $f(x,y)$ above blue annulus

$\cdot$ area of blue annulus

$P(R \in dr) = f_R(r)\,dr$

$$= c^2 e^{-\frac{1}{2}r^2} \cdot (2\pi r\,dr)$$

$$= c^2 2\pi\, r e^{-\frac{1}{2}r^2} dr$$

Rayleigh Density

$$\Rightarrow c^2 2\pi = 1 \Rightarrow c = \frac{1}{\sqrt{2\pi}}$$

$$1 = \int_{r=0}^{r=\infty} P(R \in dr) = c^2 2\pi \underbrace{\int_0^\infty r e^{-\frac{1}{2}r^2} dr}_{=1}$$

$$\Rightarrow c^2 \cdot 2\pi = 1$$

$$\Rightarrow c = \frac{1}{\sqrt{2\pi}}$$

Conclusions

1. For $X, Y \overset{iid}{\sim} N(0,1)$ and $T \sim Exp(\frac{1}{2})$

$R = \sqrt{X^2+Y^2}$ and $R = \sqrt{T}$

are both the Rayleigh distribution

little phi

2. $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ is density of the standard normal.

(i.e. $c = \frac{1}{\sqrt{2\pi}}$)

— see end of lecture notes

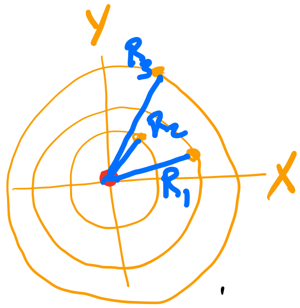
3. $E(X) = 0$ and $SD(X) = 1$

ex Suppose 3 shots are fired at a target. Assume for each shot, both X and Y are standard normals. Let W be the closest distance among 3 shots to the bullseye. Find $f_W(w)$

distance to Bullseye

$$R = \sqrt{X^2 + Y^2}$$

Picture



$$W = \min(R_1, R_2, R_3)$$

where $R_1, R_2, R_3 \overset{iid}{\sim}$ Ray

$$F_{R_1}(r) = 1 - e^{-\frac{1}{2}r^2}$$

$$P(W > w) = P(R_1 > w, R_2 > w, R_3 > w)$$

$$= \left(P(R_1 > w)\right)^3 = \left(e^{-\frac{1}{2}w^2}\right)^3 = e^{-\frac{3}{2}w^2}$$

$$F(w) = 1 - e^{-3/2 w^2}$$

$$f(w) = \frac{d}{dw} F(w) = \boxed{3we^{-3/2 w^2}, \quad w > 0}$$

Note: You could do this using Rayleigh ordered statistics since W is the min of 3 Rayleighs.

Appendix

Expectation $E(g(X,Y))$

ex

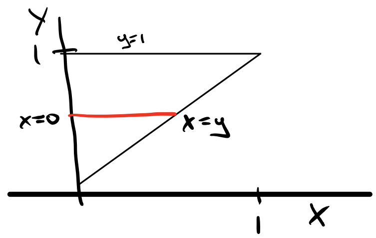
$$E(g(x,y)) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} g(x,y) f(x,y)\,dx\,dy$$

joint density

$$f(x,y) = \begin{cases} 30\,(y-x)^4 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

$X = U_{(1)}$

$Y = U_{(6)}$



Find

$$E(Y) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} y\, f(x,y)\,dx\,dy$$

$g(x,y) = Y$

$$= \int_{y=0}^{y=1} y \left[\int_{x=-\infty}^{x=\infty} f(x,y)\,dx\right] dy$$

$f_Y(y) = 6y^5$

shown this is lecture 29

$$= 6\int_{y=0}^{y=1} y^6\,dy = \frac{6y^7}{7}\Big|_0^1 = \boxed{6/7}$$

Appendix

Claim Let $X \sim N(0,1)$. We show that $E(X)=0$ and $SD(X)=1$.

Let $X \sim N(0,1)$

Show $E(X)=0$

$\phi(x)$ is symmetric about zero so all we have to show is that

$E(X)$ converges absolutely.

(i.e $E(|X|) < \infty$).

$$E(|X|) = \int_{-\infty}^{\infty} |x| \phi(x)\,dx$$

$$= 2\int_{0}^{\infty} x\phi(x)\,dx$$

$$= 2\int_{0}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi}} \underbrace{\int_{0}^{\infty} x e^{-\frac{x^2}{2}}\,dx}_{=1}$$

Rayleigh density

$$= \sqrt{\frac{2}{\pi}} < \infty.$$

$\Rightarrow E(X)=0$ since $\phi(x)$ is semetric around zero

<u>Next</u> we show $SD(X) = 1$:

We know $X^2 + Y^2 \sim Exp\left(\frac{1}{2}\right)$ from above

$\frac{1}{2} = \lambda$

so $E(X^2+Y^2) = \frac{1}{\frac{1}{2}} = 2$

λ rate of X^2+Y^2

$$\Rightarrow E(X^2) + \underbrace{E(Y^2)}_{= E(X^2)} = 2$$

$\Rightarrow 2E(X^2) = 2 \Rightarrow E(X^2) = 1$

but $SD(X) = \sqrt{E(X^2) - E(X)^2}$

$= \sqrt{1 - 0} = 1$ ✓