Stat 134 lec 5

warm up

Halloween will be here before you know it and the children in your neighborhood will come trick-or-treating (that is, they will come to your door and demand candy). Suppose there are 20 children in your neighborhood and 30 houses (one of which is yours). Each child independently chooses 10 houses at random without replacement to visit.

~~a What is the probability that a specific child will visit your house?~~

b What is the probability that exactly 10 children visit your house?

$n = 20$
$K = 10$
$P = \boxed{\frac{1}{3}}$

$P = \frac{\binom{29}{9}}{\binom{30}{10}} = \frac{1}{3}$

$P(K=10) = \binom{20}{10}\left(\frac{1}{3}\right)^{10}\left(\frac{2}{3}\right)^{10}$

$\binom{20}{10} = \frac{20!}{10!\,10!}$

$\binom{n}{K}$

$\underline{|abcd|}$

draw 2 letters w/o replacement what is chance you get d.

$\binom{4}{2} = 6$ pairs,

$\binom{3}{1} = 3$ pairs that include d.

$\frac{\binom{3}{1}}{\binom{4}{2}} = \frac{3}{6} = \frac{1}{2}$

Announcement: Quiz 1 covers Sec 1.1 – 1.6 and 2.1
3 or 4 questions in 45 minutes.

Last time

Sec 2.1 The Binomial Distribution

A Bernoulli trial has 2 outcomes, success (Prob p) and failure (Prob $q = 1 - p$).
(think of tossing a coin having prob p of landing head)

n independent Bernoulli trials, each with prob p of success, has a Binomial distribution written $\text{Bin}(n, p)$.

For $k = 0, 1, 2, \ldots, n$ the Binomial formula says

$$P(k) = \binom{n}{k} p^k q^{n-k}$$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$P(k)$: the prob of getting exactly k out of n successes.

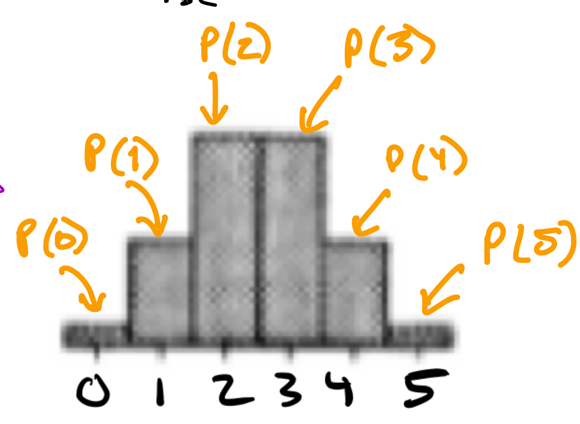
eg $n = 5$, $p = 1/2$

$P(0) = \binom{5}{0}(1/2)^0(1/2)^5 = 1/32$
$P(1) = \binom{5}{1}(1/2)^1(1/2)^4 = 5/32$
$P(2) = \binom{5}{2}(1/2)^2(1/2)^3 = 10/32$
$P(3) = 10/32$
$P(4) = 5/32$
$P(5) = 1/32$

1	1	0	0	0
1	0	1	0	0
1	0	0	1	0
1	0	0	0	1
0	1	1	0	0
0	1	0	1	0
0	1	0	0	1
0	0	1	1	0
0	0	1	0	1
0	0	0	1	1

number of arrangements of two 1s in 5 slots

$$\binom{5}{2} = \frac{5!}{2! \cdot 3!} = 10$$



Stats 134

Chapter 2 Wednesday January 30 2019

1. Ten cards are dealt off the top of a well shuffled deck. The binominal formula doesn't apply to find the chance of getting exactly three diamonds because:

   a The probability of a trial being successful changes

   b The trials aren't independent

   c There isn't a fixed number of trials

   d more than one of the above

b	only the conditional probability is changing each card turn, and not the unconditional probability. with Bernoulli distribution we use the unconditional.

c	We don't know the number of trials

d	a and b are correct: the probability of getting a Diamond changes as each of the 10 cards is dealt, therefore the trials are not independent

Compare with:

. A well shuffled deck is cut in half so there are 7 ~~aces~~ diamonds in the first half deck and 6 ~~aces~~ diamonds in the second half deck. Five cards are dealt off the top of one half deck and five cards are dealt off the top of the other half deck. The binominal formula doesn't apply to find the chance of getting exactly three diamonds total because:

a The probability of a trial being successful changes

b The trials aren't independent

c There isn't a fixed number of trials

d more than one of the above

Today

1. Finish sec 2.1 Binomial distribution
2. Start sec 2.2 Normal approximation to the binomial.

① Binomial Dist The mode and mean

What does the binomial distribution look like for different n, p?

(most likely outcome)

Def^n (Mode)
The mode of the binomial distribution is the most likely outcomes (i.e. the k such that $P(k) = \binom{n}{k} p^k (1-p)^{n-k}$ is largest)

* ← Proof at end of notes using $\frac{P(k)}{P(k-1)}$, the consecutive odds ratio,

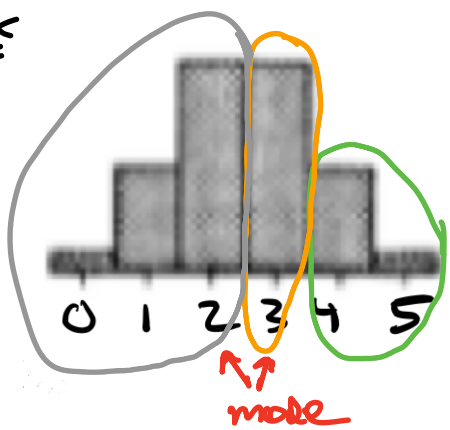
Thm For $k \in \{1, 2, \ldots, n\}$

given by Binomial Formula,

- $k < np + p$ iff $P(k-1) < P(k)$
- $k = np + p$ iff $P(k-1) = P(k)$
- $k > np + p$ iff $P(k-1) > P(k)$

Picture

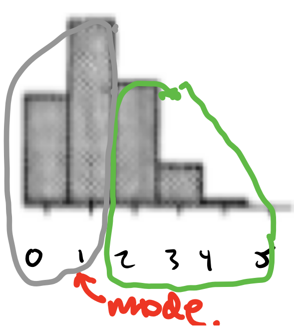
ex



$n = 5$
$k = 1, 2, 3, 4, 5$
$p = 1/2$

$np + p = 5(1/2) + 1/2 = \boxed{3}$

ex



ex $n = 5$, $p = 1/4$, $k = 1, 2, 3, 4, 5$

$np + p = 5 \cdot \frac{1}{4} + \frac{1}{4} = \boxed{1.5}$

The mode for the binomial distribution has 2 cases:

$$\text{mode} = \begin{cases} m & \text{if } np+p \notin \mathbb{Z} \\ m-1, m & \text{if } np+p \in \mathbb{Z} \end{cases} \quad \text{where } m = \lfloor np+p \rfloor$$

integers.

the integer part of $np+p$

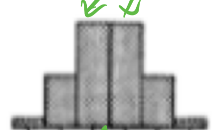
The mode is a measure of the center of the data. However, the true center of your data is the expectation (i.e the mean).

Fact The expected (mean) number of success is $\boxed{\mu = np}$

shown in chap 3

This isn't usually an integer

ex $n=5$, $p=1/2$ $\quad \mu = np = 5/2$



Given $0<p<1$, if the mean is an integer is it a mode?

Yes, $np \in \mathbb{Z} \Rightarrow np+p \notin \mathbb{Z} \Rightarrow$ there is a single mode

$m = \lfloor np+p \rfloor = np$

the mean,

Fact the average spread around the mean (standard deviation) is $\sigma = \sqrt{npq}$ where $q = 1-p$

shown in Chap 3

Notice that the spread around the mean gets larger as n gets bigger,

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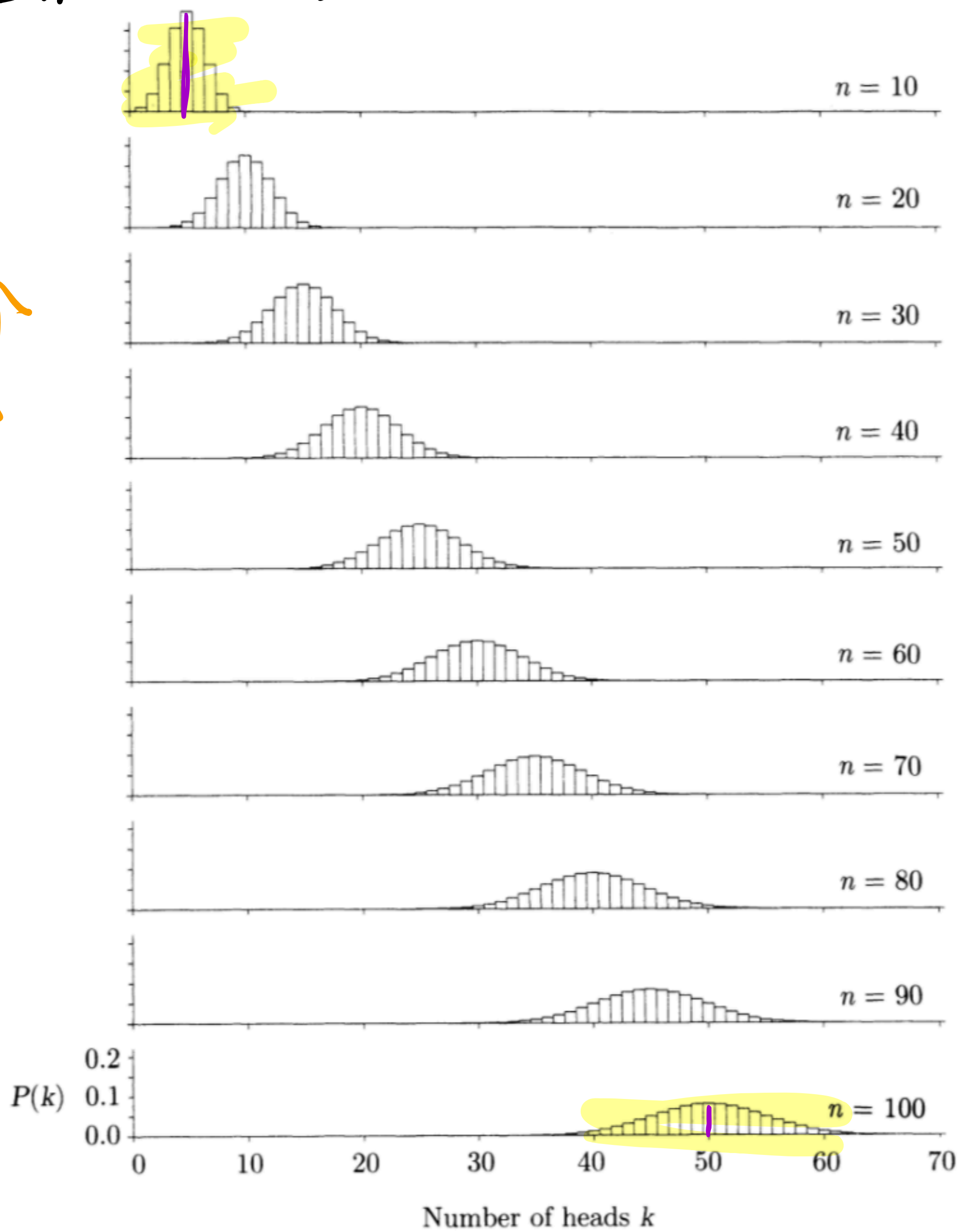
Binomial $(n, \frac{1}{2})$ $\sigma = \sqrt{npq} = \sqrt{n \frac{1}{2} \cdot \frac{1}{2}} = \frac{\sqrt{n}}{2}$

Notice spread ↑ as n ↑



Law of averages says
if X = # heads in n fair coin tosses

$\frac{X}{n} \longrightarrow p = 1/2$.

← fraction of heads in n tosses

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Stat 134

Chapter 2

1. A fair coin is tossed, and you win a dollar if there are exactly 50% heads. Which is better?

   a 10 tosses

   b 100 tosses

As n increases the spread increases, and the probability of getting any particular value goes down.

The binomial formula says

$P(5) = \binom{10}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5 = .246$

$P(50) = \binom{100}{50}\left(\frac{1}{2}\right)^{50}\left(\frac{1}{2}\right)^{50} = .08$

This doesn't violate the law of averages:

Out of 100 tosses you will get on

SD of avg $\approx .03$ → average mostly between [.47, .53]

⇒ mostly get 47, 48, 49, 50, 51, 52, 53 heads. Hard to get exactly 50

SD of avg $\approx .1$ → Out of 10 tosses you will get on average mostly between [.4, .6]

⇒ mostly get 4, 5, 6 heads. Easier to get exactly 5

② sec 2.2 The normal distribution

The normal curve is $y = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

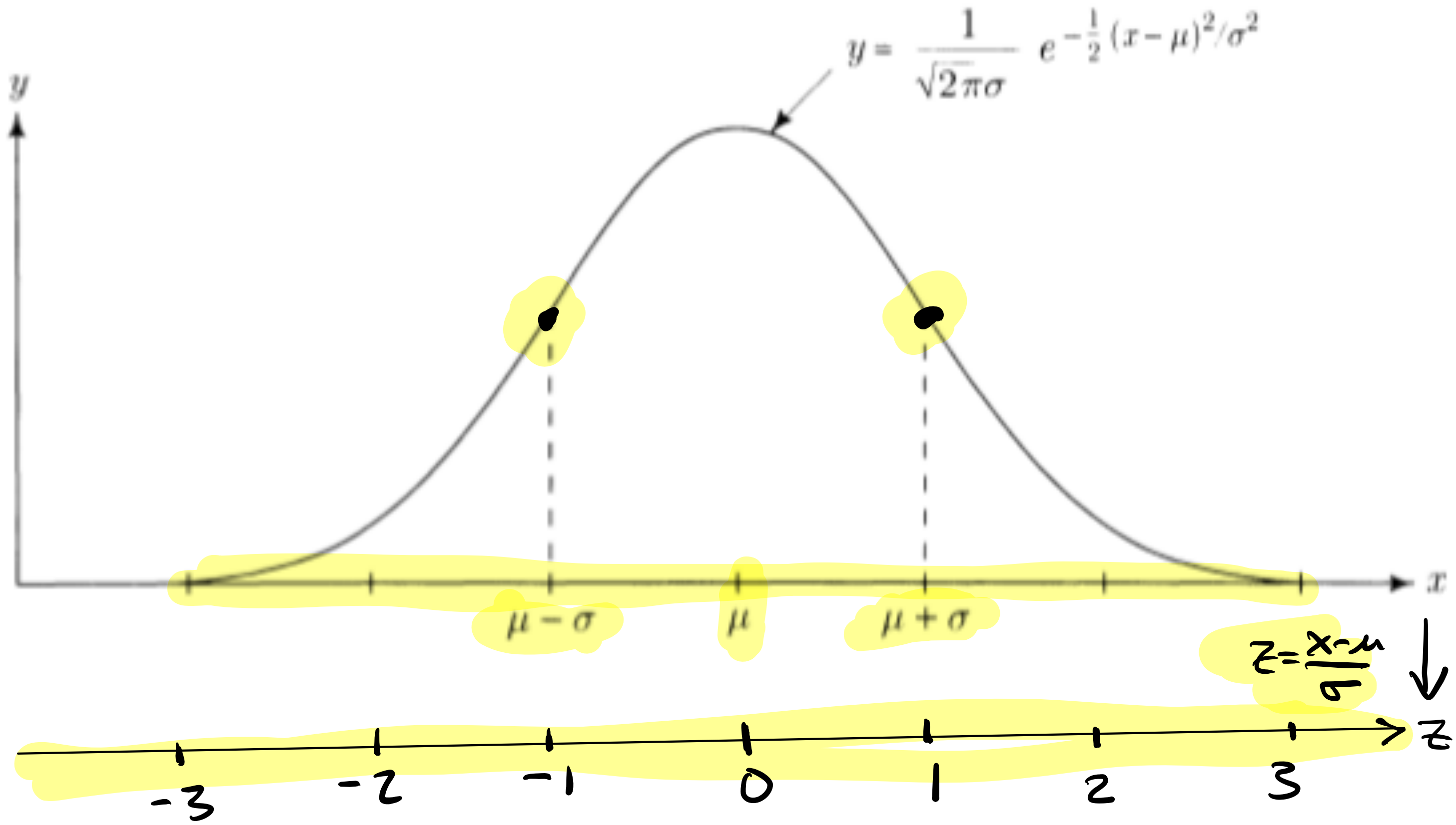
Notice :

① two param μ = mean
σ = std dev

② inflection pts $\mu \pm \sigma$

③ almost all data between $\mu \pm 3\sigma$

P93 FIGURE 1. The normal curve.



To find the area under the curve it is convenient to make a change of coordinates

$$z = \frac{x-\mu}{\sigma}$$

This makes $\mu = 0$ and $\sigma = 1$

density function

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$ std normal curve

<u>Define</u> cummulative distribution function (cdf)

as $\Phi(z) = \int_{-\infty}^{z} \phi(t)\,dt$ ← area between $-\infty$ and z

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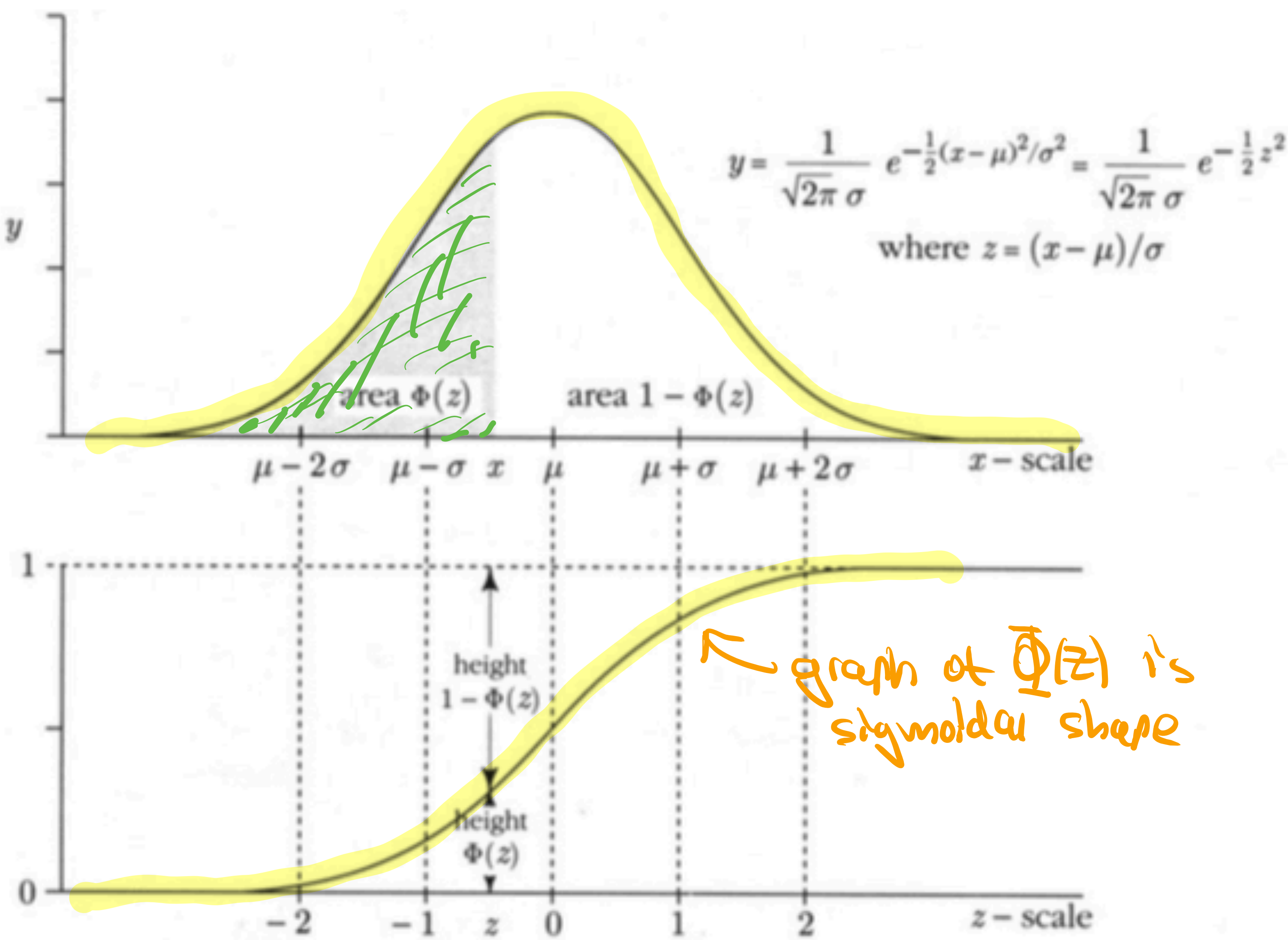
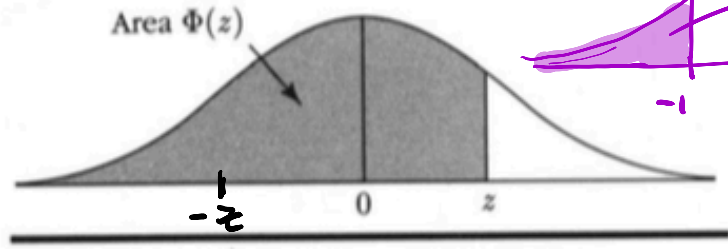


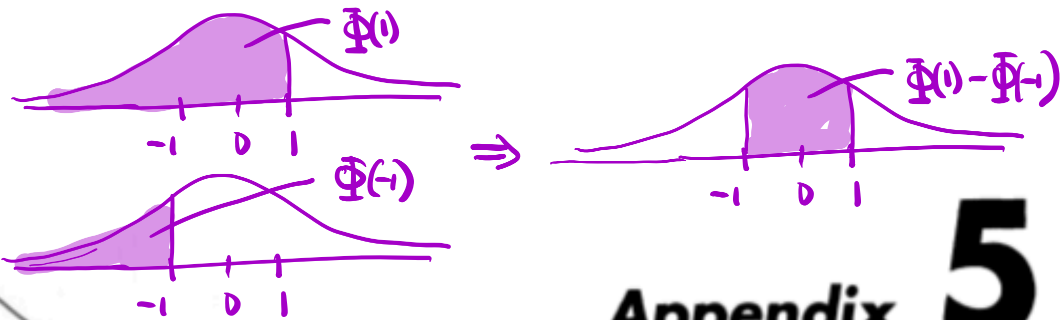
← graph of $\Phi(z)$ is sigmoidal shape

We cant solve integral $\Phi(z)$ but instead use a look up table.

Notice

$\Phi(-z) = 1 - \Phi(z)$





Appendix 5

Normal Table

Table shows values of $\Phi(z)$ for z from 0 to 3.59 by steps of .01. Example: to find $\Phi(1.23)$, look in row 1.2 and column .03 to find $\Phi(1.2 + .03) = \Phi(1.23) = .8907$. Use $\Phi(z) = 1 - \Phi(-z)$ for negative z.

	.0	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998

Find area between 1 and -1 in std normal curve:

$\Phi(1) - \Phi(-1)$

$= .8413 - (1 - .8413)$

$= \boxed{.68}$

Find area between 2 and -2.

Similarly $\to$ .95

Find area between 3 and -3.

.997

This is known as the emphirical rule,

Appendix

Thm

① $k < np + p$ iff $P(k-1) < P(k)$

② $k > np + p$ iff $P(k-1) > P(k)$

③ $k = np + p$ iff $P(k-1) = P(k)$

Pf/

First note that $$\frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{\frac{\cancel{n!}}{k!(n-k)!}}{\frac{\cancel{n!}}{(k-1)!(n-k+1)!}} = \boxed{\frac{n-k+1}{k}}$$

Called the consecutive odds ratio

binomial formula

$$\frac{P(k)}{P(k-1)} = \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\binom{n}{k-1} p^{k-1} (1-p)^{n-k+1}} = \boxed{\frac{n-k+1}{k} \cdot \frac{p}{1-p}}$$

$$P(k-1) < P(k)$$

$$\iff 1 < \frac{P(k)}{P(k-1)}$$

$$\iff 1 < \frac{n-k+1}{k} \cdot \frac{p}{1-p}$$

$$\iff k(1-p) < (n-k+1)p$$

$$\iff k - \cancel{kp} < np - \cancel{pk} + p$$

$$\iff k < np + p$$

so $P(k-1) < P(k) \iff k < np + p$

similarly for $>$ or $=$ instead of $<$ $\square$