Stat 134 lec 9

Warmup

(X_1, X_2) has joint distribution:

	$X_2 = 0$	$X_2 = 1$	
$X_1 = 0$	$\frac{5}{36}$	$\frac{25}{36}$	$5/6$
$X_1 = 1$	$\frac{1}{36}$	$\frac{5}{36}$	$1/6$
	$1/6$	$5/6$	

Is X_1, X_2 independent?

check

$P(x,y) = P(x)P(y)$ ✓

$\Rightarrow x, y$ independent

Note the entries in the cells are

$P(X=x, Y=y)$ not $P(X=x \mid Y=y)$

To find $P(X=x \mid Y=y)$ use Bayes rule

$$= \frac{P(X=x, Y=y)}{P(Y=y)}$$

Last time

Sec 3.1 Random Variables

The event $(X=x, Y=y)$ is the intersection of events $X=x$ and $Y=y$. ← sometimes written (x,y)

The probability X and Y satisfies some condition (i.e. $P(X+Y=s)$) is the sum of $P(x,y)$ that satisfy that condition.

ex $$P(X+Y=s) = \sum_{(x,y):x+y=s} P(x,y) = \sum_{\text{all } x} P(x, s-x)$$

Independence of (X,Y,Z) means

$$P(X=x, Y=y, Z=z) = P(X=x)P(Y=y)P(Z=z)$$ for all $x \in X$, $y \in Y$, $z \in Z$.

b	They cannot be independent therefore a,c,d cannot be correct

d	These are all true, b) iff a) and c) if a)

stat 134 concept test

The joint distribution of X and Y is drawn below:

	3/8	1/2	1/8	P(x) \ P(y)
1	1/4	1/3	1/12	2/3
0	1/8	1/6	1/24	1/3
Y \ X	0	1	2	

a) X and Y are independent

b) If we divide both rows by their marginal probability we get the same answer.

c) $P(X = x|Y = 0) = P(X = x|Y = 1)$

d) All of the above

Today

① Sec 3.1 Sums of independent Poissons is Poisson

② Sec 3.2 Expectation of a RV.

① Sum of independent Poisson is Poisson

informal argument:

$$X_1 \sim \text{Bin}\left(1000, \tfrac{1}{1000}\right) \approx \text{Pois}(1)$$

$$X_2 \sim \text{Bin}\left(2000, \tfrac{1}{1000}\right) \approx \text{Pois}(2)$$

indep

$$X_1 + X_2 \sim \,? \quad \text{Bin}\left(3000, \tfrac{1}{1000}\right) \approx \text{Pois}(3)$$

proven in appendix to these notes

Claim If $X \sim \text{Pois}(\mu)$ and $Y \sim \text{Pois}(\lambda)$ are independent then

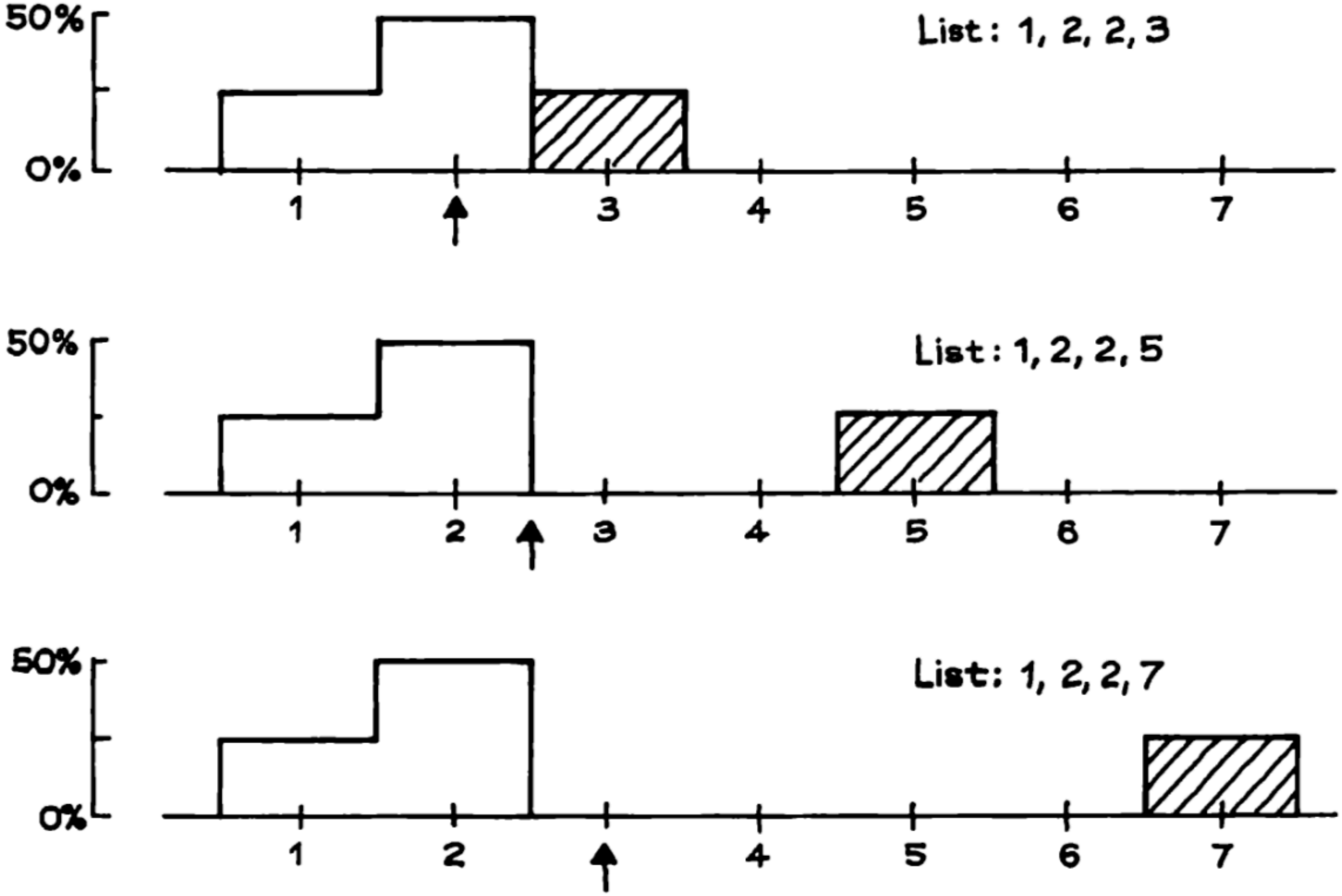
$$S = X + Y \sim \text{Pois}(\mu + \lambda).$$

2

sec 3.2 Expectation

$$E(X) = \sum_{x \in X} x \cdot P(X=x)$$

$$E(X) = 1 \cdot \tfrac{1}{4} + 2 \cdot \tfrac{1}{2} + 3 \cdot \tfrac{1}{4} = 2$$



$$E(X) = \sum_{x \in X} x \cdot P(X=x)$$

Properties of Expectation — P167 Pitman

① $E(c) = c$

② $E(X+Y) = E(X) + E(Y)$ (X, Y <u>dont</u> need to be independent)

③ $E(aX+b) = aE(X) + b$

Indicators

An indicator is a RV that has only 2 values 1 (w/ prob p) and 0 (with prob $1-p$).

$$I = \begin{cases} 1 & \text{prob } p \\ 0 & \text{prob } 1-p \end{cases}$$ — same as a Bernoulli p trial.

$$E(I) = 1 \cdot p + 0 \cdot (1-p) = p$$

RV that are counts can often be written as a sum of indicators.

ex $X \sim Bin(n, p)$

\# successes in n Bernoulli p trials.

ex X = # heads in n flips of p coin

$X = I_1 + I_2 + \dots + I_n$

where $I_j = \begin{cases} 1 & \text{if } j^{th} \text{ trial success} \\ 0 & \text{else} \end{cases}$ p

$$E(X) = \underset{p}{\underset{\|}{E(I_1)}} + \dots + \underset{p}{\underset{\|}{E(I_n)}} = \boxed{np}$$

indicators are independent since trials are indep.

Ex X = # aces in a poker hand from a deck of cards

$X \sim HG(n=5, N=52, G=4)$

a) what are the range of values of X?

$0, 1, 2, 3, 4$

b) write X as a sum of indicators

$X = I_1 + I_2 + I_3 + I_4 + I_5$

c) How is I_2 defined?

$p = 4/52$

$$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ card is an ace} \\ 0 & \text{else.} \end{cases}$$

d) Find $E(I_2)$

$= 4/52$

e) Find $E(X) = E(I_1) + E(I_2) + E(I_3) + E(I_4) + E(I_5)$

$E(I_1) = 4/52$, $E(I_2) = 4/52$, $E(I_3) = 4/52$, $E(I_4) = 4/52$

$= 5(4/52)$

Another more complicated solution:

Note

You may define $I_2 = \begin{cases} 1 & \text{if get 2 aces} \\ 0 & \text{else} \end{cases}$

so

$$X = I_1 + 2I_2 + 3I_3 + 4I_4$$

This is also correct but more complicated.

$$E(I_1) = \frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}} \qquad E(I_3) = \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}$$

$$E(I_2) = \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} \qquad E(I_4) = \frac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$$

so $E(X) = \frac{1}{\binom{52}{5}} \left[\binom{4}{1}\binom{48}{4} + 2\cdot\binom{4}{2}\binom{48}{3} + 3\cdot\binom{4}{3}\binom{48}{2} + 4\binom{4}{4}\binom{48}{1} \right]$

$= 5\cdot\left(\frac{4}{52}\right)$ ← I checked this in R

Note $E(X) = 1\cdot P(X=1) + 2\cdot P(X=2) + 3\cdot P(X=3) + 4\cdot P(X=4$

ex

A drawer contains s black socks and s white socks (s> 0). I pull two socks out at random without replacement and call that my first pair. Then I pull out two socks at random without replacement and call that my second pair. I proceed in this way until I have s pairs and the drawer is empty. Find the expected number of pairs in which two socks are different colors.

X = # of mismatched pairs (out of s)

$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ pair mismatched} \\ 0 & \text{else} \end{cases}$

$X = I_1 + \dots + I_s$

Finish problem ...

Appendix

Claim If $X \sim Pois(\mu)$ and $Y \sim Pois(\lambda)$ are independent then

$S = X + Y \sim Pois(\mu + \lambda)$.

To prove this you need to know 2 facts:

Recall binomial theorem

$$(a+b)^3 = \binom{3}{3} a^3 b^0 + \binom{3}{2} a^2 b^1 + \binom{3}{1} a^1 b^2 + \binom{3}{0} a^0 b^3$$
$$= a^3 + 3a^2 b + 3ab^2 + b^3$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Recall $X \sim Pois(\mu)$

$$P(X=k) = \frac{e^{-\mu} \mu^k}{k!}$$

Pf/

$$P(S=s) \overset{\text{addition rule}}{=} P(X=0, Y=s) + P(X=1, Y=s-1) + \ldots P(X=s, Y=0)$$

$$\overset{\text{summation notation}}{=} \sum_{k=0}^{s} P(X=k, Y=s-k)$$

$$\overset{\text{independence of } X \text{ and } Y}{=} \sum_{k=0}^{s} P(X=k) P(Y=s-k)$$

Poisson formula

$$= \sum_{k=0}^{s} \frac{e^{-\mu}\mu^{k}}{k!} \cdot \frac{e^{-\lambda}\lambda^{s-k}}{(s-k)!}$$

$\frac{s!}{s!} = 1$

$$= e^{-(\lambda+\mu)} \frac{1}{s!} \sum_{k=0}^{s} \frac{s!}{k!(s-k)!} \mu^{k} \lambda^{s-k}$$

binomial thm

$$= e^{-(\lambda+\mu)} \frac{1}{s!} (\mu+\lambda)^{s}$$

$$\Rightarrow S \sim \mathrm{Pois}(\mu+\lambda).$$

poisson formula.

$\square$