STAT 134 - Instructor: Adam Lucas

Midterm 2

Friday, November 20, 2020

Print your name: ________________________________

SID Number: ________________________________

Exam Information and Instructions:

- You will have 48 hours to take this exam. Open book/notes but no internet resources outside of the Stat134.org website. You are allowed to use a caluculator.
- We will be using Gradescope to grade this exam. Write any work you want graded on the front of each page, in the space below each question. Additionally, write your SID number in the top right corner on every page.
- Provide calculations and reasoning in every answer.
- Unless stated otherwise, you may leave answers as unsimplified numerical and algebraic expressions, and in terms of the Normal c.d.f. Φ. Finite sums are fine, but simplify any infinite sums.

I certify that all materials in the enclosed exam are my own original work and I have not violated the UC Berkeley honor code.

Sign your name: ________________________________

GOOD LUCK!

1. Honor Code

   (1 pt) Circle all choices that violate the rules of this test or the UC Berkeley Honor Code?

   (a) post or read on Chegg or an online forum

   (b) use a calculator

   (c) leaving your answer unsimplified

   (d) use your notes, or textbook or a cheat sheet

   (e) communicate with non-staff about the test until we communicate with you that everyone has taken the test.

2. New MGF

Sankar and Zhiyi are unsatisfied with the moment-generating function, and invent a new function called the S-function. The S-function of a random variable X is defined as:

$$S_X(t) = M_{\log(X)}(t)$$

Where $M_{\log(X)}(t)$ is the moment-generating function of $\log(X)$.

Now let X, Y be independent and identically distributed as per the following density:

$$f(z) = \frac{1}{z^2}, 1 \leq z < \infty$$

(a) (5 pts) Show that $S_{\mathbf{X}}(t) = \frac{1}{1-t}$, $t < 1$.
Hint: You do not need to do a change of variables for this.

(b) (5 pts) Find $S_{\mathbf{XY}}(t)$, which is the S-function of the product of X and Y, and make sure to define where it is finite.

3. Drunken Ant

Suppose that at each timestep t, a drunken ant moves randomly in the x and y directions such that the movements are independent of one another across direction and time. Let $X_t, Y_t \sim \mathcal{N}\left(0, \sigma^2\right)$.

(a) (3 pts) Find the distribution of $\sum_{t=1}^{n} X_t$.

(b) (5 pts) Find the probability density function for,

$$\sqrt{\left(\sum_{t=1}^{n} X_t\right)^2 + \left(\sum_{t=1}^{n} Y_t\right)^2},$$

which is the distance between the ant's starting point and its position at time step n.

(c) (2 pts) What does this density look like as the number of time steps n gets larger, and what does this say about where the ant may end up in the distant future? (A sound logical argument suffices. No calculations needed.)

4. (9 pts) Out of Battery

After learning of the new P/NP grading options, you think about the chance of being able to (separately) call your two closest friends to share the news, before your phone battery dies.

Assume that:

- the calls last i.i.d. $\text{Exp}(\lambda)$ amounts of time;
- the phone battery has an $\text{Exp}(\mu)$ amount of life remaining when you begin making calls; and
- the call durations and the battery life are independent.

You quickly calculate the probability that you complete both calls before your phone dies, as follows.

Let X be the battery life and Y_1 and Y_2 the durations of the first two calls.

$$\begin{aligned}
\mathbb{P}(X \geq Y_1 + Y_2) &= \mathbb{P}(X \geq Y_1 + Y_2 | X \geq Y_1) \cdot \mathbb{P}(X \geq Y_1) \\
&= \mathbb{P}(X \geq Y_2) \cdot \mathbb{P}(X \geq Y_1) \\
&= \mathbb{P}(X \geq Y_1)^2 \\
&= \left(\frac{\lambda}{\mu + \lambda}\right)^2.
\end{aligned}$$

Is your derivation correct? If yes, justify each step. If not say what is wrong.

5. Two Uniforms

   Suppose there are two i.i.d random variables $U_1, U_2 \sim \text{Uniform}(0,1)$. Let $X = \max\{U_1, U_2\}$, $Y = \min\{U_1, U_2\}$.

   (a) (5 pts) Derive the joint density of X and Y and show your work.

   (b) (5 pts) Prove via the convolution formula that the density of $Z = X - Y$ is Beta distribution, and provide the parameters.

6. Family Trip

   (a) (5 pts) A family is getting ready for their trip to Yosemite. Each person is in their room, packing their bags. For each person, the time it takes them to pack their bag is exponentially distributed and independent of the time it takes any other person. On average, it takes each parent 1 hour and each child 2 hours to get ready. In a family with 2 parents and 4 children, what is the probability that it takes the family more than 2 hours to get ready?

   (b) (5 pts) After setting up the tents the family wants to rent bicycles. The rental place does only have one bicycle when they get there. Assume that the bikes being returned follow a Poisson process with rate $\frac{1}{10}$ (measured in minutes) and that nobody else is waiting for bikes. Give an expression for the probability that the family will have to wait more than 30 minutes before they can start their biking tour. (No need to simplify!)

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Scratch Paper

Discrete

name and range	$P(k) = P(X = k)$ for $k \in$ range	mean	variance
uniform on $\{a, a+1, \ldots, b\}$	$\dfrac{1}{b-a+1}$	$\dfrac{a+b}{2}$	$\dfrac{(b-a+1)^2-1}{12}$
Bernoulli (p) on $\{0, 1\}$	$P(1) = p$; $P(0) = 1 - p$	p	$p(1-p)$
binomial (n, p) on $\{0, 1, \ldots, n\}$	$\binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$
Poisson (μ) on $\{0, 1, 2, \ldots\}$	$\dfrac{e^{-\mu}\mu^k}{k!}$	μ	μ
hypergeometric (n, N, G) on $\{0, \ldots, n\}$	$\dfrac{\binom{G}{k}\binom{N-G}{n-k}}{\binom{N}{n}}$	$\dfrac{nG}{N}$	$n\left(\dfrac{G}{N}\right)\left(\dfrac{N-G}{N}\right)\left(\dfrac{N-n}{N-1}\right)$
geometric (p) on $\{1, 2, 3 \ldots\}$	$(1-p)^{k-1}p$	$\dfrac{1}{p}$	$\dfrac{1-p}{p^2}$
geometric (p) on $\{0, 1, 2 \ldots\}$	$(1-p)^k p$	$\dfrac{1-p}{p}$	$\dfrac{1-p}{p^2}$
negative binomial (r, p) on $\{0, 1, 2, \ldots\}$	$\binom{k+r-1}{r-1} p^r (1-p)^k$	$\dfrac{r(1-p)}{p}$	$\dfrac{r(1-p)}{p^2}$

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Continuous

† undefined.

name	range	density $f(x)$ for $x \in$ range	c.d.f. $F(x)$ for $x \in$ range	Mean	Variance
uniform (a, b)	(a, b)	$\frac{1}{b-a}$	$\frac{x-a}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
normal $(0, 1)$	$(-\infty, \infty)$	$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$	$\Phi(x)$	0	1
normal (μ, σ^2)	$(-\infty, \infty)$	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$	$\Phi\left(\frac{x-\mu}{\sigma}\right)$	μ	σ^2
exponential (λ) = gamma $(1, \lambda)$	$(0, \infty)$	$\lambda e^{-\lambda x}$	$1 - e^{-\lambda x}$	$1/\lambda$	$1/\lambda^2$
gamma (r, λ)	$(0, \infty)$	$\Gamma(r)^{-1} \lambda^r x^{r-1} e^{-\lambda x}$	$1 - e^{-\lambda x} \sum_{k=0}^{r-1} \frac{(\lambda x)^k}{k!}$ for integer r	r/λ	r/λ^2
chi-square (n) =gamma $\left(\frac{n}{2}, \frac{1}{2}\right)$	$(0, \infty)$	$\Gamma(\frac{n}{2})^{-1} (\frac{1}{2})^{\frac{n}{2}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}$	as above for $\lambda = \frac{1}{2}$, $r = \frac{n}{2}$ if n is even	n	$2n$
Rayleigh	$(0, \infty)$	$x e^{-\frac{1}{2}x^2}$	$1 - e^{-\frac{1}{2}x^2}$	$\sqrt{\frac{\pi}{2}}$	$\frac{4-\pi}{2}$
beta (r, s)	$(0, 1)$	$\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1} (1-x)^{s-1}$	see Exercise 4.6.5 for integer r and s	$\frac{r}{r+s}$	$\frac{rs}{(r+s)^2(r+s+1)}$
Cauchy	$(-\infty, \infty)$	$\frac{1}{\pi(1+x^2)}$	$\frac{1}{2} + \frac{1}{\pi} \arctan(x)$	†	†
arcsine =beta $(1/2, 1/2)$	$(0, 1)$	$\frac{1}{\pi\sqrt{x(1-x)}}$	$\frac{2}{\pi} \arcsin(\sqrt{x})$	$\frac{1}{2}$	$\frac{1}{8}$