Stat 134: Section 19

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Problem 1: Conditioning on the First Toss

Let X be the number of tosses to get heads in a coin that lands heads with probability p.

a. Argue that given the first toss is tails, the number of tosses to get heads is modeled by $1 + X^*$, where X^* and X have the same distribution.

b. Let I_1 be the indicator of whether the first toss is heads. Use part (a) and the rule $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|I_1))$ to show $\mathbb{E}(X) = 1/p$.

Problem 2

Let $X \sim$ Exponential (λ), and let $Y \sim$ Poisson (X) (that is, given $X = x$, Y follows the Pois (x) distribution).

a. Find $P(X \in dx, Y = y)$;

b. Use (a) to find the unconditional distribution of Y;

c. Given $Y = y$, what is the conditional density of X? (Hint: use Bayes' Rule).

Problem 3

Suppose that a point (X, Y) is uniformly chosen at a random from the triangle

$$\{(x,y) : x \geq 0, y \geq 0, x + y \leq 2\}.$$

a. Find a formula for $P(Y \leq y|X = x)$.

b. Find $E[Y|X = x]$.

c. FInd $Var[Y|X = x]$.

Problem 4

Define $\text{Var}[Y|X]$, the conditional variance of Y given X, to be the random variable whose value, if $(X = x)$, is the variance of the conditional distribution of Y given $X = x$. So $\text{Var}[Y|X]$ is a function of X, namely $h(X)$, where $h(x) = E[Y^2|X = x] - [E[Y|X = x]]^2$.

a. Show that $Var(Y) = E[Var(Y|X)] + Var[E(Y|X)]$.

b. Check part a. for joint distribution in Problem 3 above.

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